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MOLECULAR MAGNETISM

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BY

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UNIVERSITY OF ST. ANDREWS

I believe that we have there (in Weber's representation) the explanation of magnetism in solids.
It is interesting to see all these things come out as they do.

KELVIN

LONDON
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1929
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TO MY WIFE

HELPER AND COMRADE

NOW AS EVER

"And this word, Yet once more, signifyeth the removing of those things that are shaken, as of things that are made, that those things which cannot be shaken may remain."

"If the vision tarry, wait for it, for it will come; and it will not be late."

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PREFACE

THE foundations of every one of the great branches of Natural Philosophy have been probed beyond the merely phenomenal or molar stage of their description. They are now treated, in large part at least, in the light of molecular or even sub-atomic ideas. Upon every one of them has been cast the illumination of the two great speculative modern expansions of the older physical theory. And, in turn, it lies with them, by appeal to the decision of experiment to discriminate amongst the various forms which these speculations may assume, and to lead to the recognition of their underlying compulsions, and so to the elimination of what is arbitrary in their formulations.

Equally with all the other branches, the subject of Magnetism has shared in the general advance; and the attention of investigators, theoretical and practical alike, is now, and is increasingly, turned upon it. Optics and Chemistry in particular wait upon its developments.

But, whereas all the other subjects have their representation amongst the honours courses in Universities, and their aids to study amongst the series of textbooks, this subject has lain hitherto outside these courses, possibly for the reason that no textbook has yet appeared. Indeed, until the recent publication of Stoner's work, *Magnetism and Atomic Structure*, in Britain, and of *Le Magnetisme* in France, original papers were largely the only source of information available to the student.

It is not the aim of this book to occupy also the ground taken by these works. It is rather to furnish a textbook which may, on the one hand, serve as an accompaniment to a short University course of lectures on the subject; and, on the other, be of use as an introduction to these more detailed treatments. A very large bibliography, given by Stoner, will be advantageous to the student who desires to proceed further.

And I trust that, with whatever aid or stimulus may be found

herein, some students may be induced to adopt the subject as one in which they will become original investigators.

My thanks are specially due to members of my staff for much voluntary aid: to Dr. Forrest for reading all the chapters in manuscript, and for frequent helpful discussions; to Dr. Winifred Smith for careful proof reading and drawing of all the diagrams; and to Mr. Raymond Smart, M.A., for verification of formulæ. I have also to thank Mr. G. N. Hunter, B.Sc., for carrying out troublesome calculations, and plotting several curves by their aid.

W. PEDDIE.

July 17, 1929.

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MOLECULAR MAGNETISM

CHAPTER I

THE ELEMENTARY PHENOMENA OF MAGNETISM AND THE MAGNETIC FIELD

1. Magnetism and Electricity.—The fundamental properties of the lodestone and of ordinary steel magnets will be familiar to any who desire to undertake a study of the still more fundamental phenomena, of molecular type, which underlie these and produce them by summation of actions. Just as electricity was thought of at first as a fluid whose presence in an electrified body conferred upon that body its special electrical properties, so magnetism was regarded as a substance whose presence in a magnetized body originated the special magnetic properties which are manifested by that body. In each case the presence of electricity, or of magnetism, that is to say, the existence of the condition of electrification or of magnetization, is inferred from the existence of mutual actions between the electrified or the magnetized bodies. And the region in which these mutual influences are existent is termed the electric, or magnetic, field, respectively. Electrified or magnetized bodies which are free to move in the field are in general set in motion in definite ways. Therefore forces are acting upon them, and for that reason the field is termed a field of force. Further, the nature of these forces is such that we recognize two kinds of electricity, positive and negative respectively; and two kinds of magnetism, north and south respectively.

2. Electric and Magnetic Quantity.—Quantity of matter has been recognized from prehistoric times; and for centuries it has been "weighed" out by taking the ratio of the force which acts upon it in the earth's gravitational field to the force with

which that field acts upon a standard mass—the pound, the gramme, etc. This is what is done when the beam balance is employed. But, otherwise, if we take unit force as the force which, acting upon unit mass, chosen arbitrarily, gives rise to unit acceleration of motion, we can measure the quantity of matter by dividing the measured force by the measured acceleration. The former method is statical; the latter is kinetic, but, in it, the measurement of the force may be made statically by means, say, of the extension of a spiral spring.

The most self-contained method of measuring gravitational force would be in terms of the unit given by the gravitational attraction between two unit masses placed at unit distance apart; but with any of the usual units of mass and length its magnitude would be impossibly small for practical purposes.

Now the fact that small feebly electrified or magnetized bodies can exert mutual forces which exceed the gravitational attraction of the whole earth upon either of them, shows that these forces are very large in comparison with gravitational forces. They are, in fact, so large that the unit of electric force is conveniently defined as the force acting between two unit quantities of electricity placed at unit distance apart, or conversely, the unit of electric quantity can be taken as the quantity such that two units placed at unit distance apart will exert upon each other the (previously chosen) unit of force. This is called the *electrostatic* unit of electric quantity. Similarly, the term *magnetostatic* might be used for the magnetic unit of quantity, but the name *electro-magnetic* unit is adopted in this case.

3. The Characteristics of Electric and Magnetic Forces and Fields of Force.—The Newtonian law of gravitational action between masses is found to be applicable, in respect of its general form, to the action between electric charges and magnetic quantities, respectively, that is to say, distributions of electricity exert mutual forces which may be calculated by assuming that any two small portions of them exert upon each other a force, which is in the direction of the straight line joining them, is proportional to the amount of electricity in each, and is inversely proportional to the square of the distance between them; and so also in the case of distributions of magnetism. A fundamental difference between either of these and gravitation, is that gravita-

tional forces are attractive only. In these words, no negative masses of matter exist.

Further distinctions are that while matter cannot act so as to screen, even to the slightest extent, other matter from the influence of a gravitational field of force, it is possible, by means of suitable bodies, to screen electricity entirely from the action of an electrical field; and, by means of other material bodies, to screen it partly. The student of Junior Physics is aware that bodies of the former class are conductors, while those of the latter are insulators, of electricity. In the latter case the efficiency depends on the specific inductive capacity or dielectric power of the insulator. But, in the case of magnetic fields, it is not possible to make the screening complete. This suggests that there is no such thing as a conductor of magnetism, and points to a very fundamental distinction between electricity and magnetism inherent in their ultimate nature.

That being quite true, it has also to be remembered that modern investigation shows that an electric current flowing in a conductor, such as a metallic wire, is really a convection current. The current is maintained by the passage of electrons along the wire, carrying their electric charge with them. In this way we have justification of the old term "electric fluid"; and that to as great an extent as in the case of the flow of a material fluid consisting of discrete molecules. In the conductor of electricity the positive and negative charges are freely separable. And the conclusion to which we are compelled therefore is that, in a magnetized body, the north and south magnetisms are not freely separable. There are no separate atoms of magnetism in the sense in which we may speak of the electron as an atom of electricity.

That view is in entire agreement with the well-known facts that the north and south magnetisms in a magnet cannot be isolated by breaking the magnet into halves, and that this condition persists even when molecule is separated from molecule, say, by dissolving iron in an acid.

This condition forms the basis of the Molecular Theory of Magnetism.

4. Electric and Magnetic Induction.—The induction of electricity in a solid conductor, say by a positive charge intro-

duced into its neighbourhood, consists in the flow of electrons along the conductor towards the parts nearest this charge. This flow leaves an excess of protons in the portions of the conductor most remote from the external positive charge. The protons do not move in this case. But in a fluid conductor, such as mercury, they may move, by a process of convection, with the ion of which they form a constituent part.

When the case of an insulator is considered, the positive and negative carriers are not completely separated from each other. The electric field which acts upon them causes them to be partially separated, within the atom or molecule, in the direction in which the force acts. And this partial separation gives rise to the effective manifestation of positive and negative electricity, respectively, at the surfaces of the insulating medium.

The phenomena of magnetic induction are exactly similar to those of electric induction in an insulator. That is to say, the processes of induction in these two cases are identical as to the nature of their results. But it does not follow that the processes are identical in their mode of establishment of the results. From the analogy of the electrical case, we might presume that separate particles of north and south magnetisms exist in the atoms permanently, and that these can be partially separated within the atom by the magnetic field which acts upon it. But if the positive and negative electricities in a molecule, in the electrical case, are separate apart from the action of the field, while the line joining them has a component perpendicular to the direction of the field, the institution of the field will tend in general to orient the electric di-pole. Thus, even if the action of the field gave rise to no further increase of the electric moment through increased separation of the electric charges, a resultant electrification would be manifested. The question of the existence of one or other of these processes in the case of magnetization is a fundamental point of theory.

5. The Electric Origin of a Magnetic Field.—It is a matter of elementary knowledge now that, when electric currents flow in two neighbouring circuits, mutual forces act upon these circuits and tend to produce motion. And these forces may act either as repulsions or as attractions between the circuits. This is entirely analogous to the actions between magnets; and a

magnet, if it be placed in the neighbourhood of such a circuit, is found to be acted upon by forces just as if it were in the presence of magnets. The fields due to the currents therefore act as if they were magnetic fields; and full investigation shows that no distinction can be made between them in respect of the nature of their influences. It is therefore impossible to avoid the supposition of their essential identity.

6. Equivalent Magnets.—Now in any definite magnetic field, the magnitude and direction of the field are fixed at every

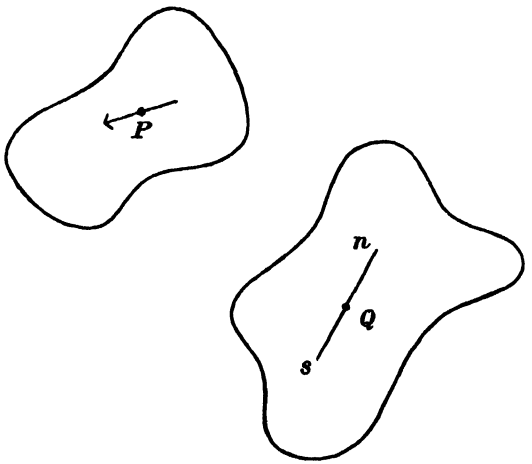

FIG. 1.

point. If we consider one point, P, alone, we can produce the given field at that point by means of a suitable magnet placed at any other point Q. For if any magnet placed at Q be directed with its north pole towards P, the field produced at P will be in the direction QP. If the south pole be towards P, the direction of the field at P will be PQ. And, if the magnet at Q be directed transversely to PQ, the field at P is also transverse to PQ, and is directed rightwards or leftwards across it according as the north pole of the magnet at Q points leftwards or rightwards. As the magnet at Q rotates, the field at P rotates correspondingly. Therefore the direction of the given field at P can be produced by means of a magnet placed at any point Q. And the magnitude of the field at P can then be produced by proper selection of the

strength of the magnet at Q. Indeed it is a matter of elementary knowledge that the magnitude of the effect at P depends on the *Magnetic Moment* only of the magnet at Q. By proper adjustment of the moment, and of the altitude and azimuth adjustments of a small magnet at any point Q, we can establish a given field at P. Also, the field varying continuously in the neighbourhood of P, the equivalence will hold throughout a small region at P.

If we now consider respectively adjacent points P' and Q', a similar adjustment could give equivalence of the actual and the simulated fields at P'. But the presence of the magnet at Q' would disturb the preceding adjustment. Yet, since six data are fixed by the given states of the field at P and P'; and since six independent variables are at disposal in the moments, azimuths, and altitudes of the small magnets at Q and Q'; it is possible to satisfy the conditions of equivalence of the fields at the two points simultaneously. The process is obviously extensible, throughout finite volumes.

There is no necessity for the condition that Q and Q' should be adjacent. Thus the order of the building up of the parts of the magnet in the Q space has no effect on the result. Evidently also the Q distribution can be made practically continuous. Thus we have an equivalent magnet which would give the P field. The mode of construction shows that there is an infinity of ways in which it may be done.

The little magnet at P, occupying the volume dv, and possessing necessarily a definite moment dM, satisfies the condition

$$Idv = dM$$

where I is the magnetic moment per unit volume. In particular, if I be a given constant, the volume throughout which it remains so must have the value $v = M/I$. In general, the distribution of values of I is arbitrary, and the Q space may exhibit both surface and volume magnetization.

7. Equivalent Electric Circuits.—Since electric currents produce magnetic fields, it should similarly be possible to find a distribution of currents, which will give rise to the given field P. A given small closed electric circuit produces a definite field. Therefore the process of the preceding section may be carried out with small electric circuits instead of small magnets.

A small magnet equivalent to a given small circuit must have a definite magnetic moment so as to produce a field of given intensity at a given point. But, as in the case of the little equivalent magnet in the preceding section, there is, apart from orientation, no other condition imposed on it.

In both cases, if we make the little magnets, or the little circuits, sufficiently small and sufficiently numerous, we can make the field produced by them, at corresponding closely packed points in the P space, exactly accurate. Therefore we can simulate the actual field in the P space as exactly as we may desire. We can make the little equivalent magnets or circuits fill the Q space continuously if we desire. But, from the point of view of molecular theory, our object is to retain the separate little bodies as models of the molecules or atoms of an actual material magnet which would produce the given field throughout the region P. If we regard these bodies hypothetically as little magnets we leave the nature of magnetism unexplained, but may nevertheless find it possible to explain on that hypothesis various properties of magnetized masses. If we regard them as little bodies within which electric currents flow, we add a further hypothesis, but attain to greater unity of idea, for then all magnetic phenomena are attributed to electric origins.

This simplification was introduced by Ampère, and is known as Ampère's Hypothesis of Molecular or Atomic Currents. It was adopted by Weber in his initiation of the modern Theory of Molecular Magnetism, and more recent investigation has added nothing more, to the foundations of Weber's Theory, than the confirmation of Ampère's postulate, with the additional information that the molecular currents are convection currents caused by the orbital motion of the exterior electrons of the atoms.

8. The Equivalent Magnetic Moment.—The size of the orbits of the electrons in the atoms being very small in comparison with ordinary distances, while the speed of the revolutions of the electrons is very great, the convection current practically acts as a steady current would act, so far as the magnetic effect at points outside the magnet is concerned.

In the case of a steady current i flowing in a small circuit of area a, the magnetic moment is proportional to i and to a. For,

a second similar and similarly situated circuit of that same area, and carrying an equal current, closely superposed upon the first, produces by symmetry, at distant points, the same magnetic effect as the first. But the effect of the two currents is, at the distant point, indistinguishable from that of a double current flowing through the first circuit alone. By similar reasoning we conclude that the current is proportional to i. Again, a second equal and similarly orientated circuit, carrying an equal current, and placed very near the first, produces at a distant point an equal effect, and so on. Thus we conclude that the effect at a distant point is also proportional to a. In both cases the deduction from symmetry is confirmed. Thus we get for M, the magnetic moment of the small circuit, the expression

$$M = kia$$

where k is a constant. In the electromagnetic system of units, k becomes unity, and we get

$$M = ia$$

Now, if this current is a convection current, due to the passage of a single electron of charge e, circulating in an orbit of area a, with high frequency n, the charge e passes any one point of the orbit n times per second and so the total amount of electricity transferred past the given point in the path is ne per second. But this is, by definition, the strength of the equivalent current. Hence, in this case, the expression becomes

$$M = nea$$

e being measured in electromagnetic units.

If N be the number of co-directed orbits of the same area, in unit volume of the magnetized body, the magnetic moment per unit volume of the substance (which is equal to the quantity of magnetism, I, per unit area of a surface taken transverse to the direction of magnetization) is

$$I = nNea$$

In general it may be expressed as

$$I = \Sigma \, . \, nea$$

where the summation is taken per unit volume.

9. The Magnetic Potential and the Force Components.—The magnetic potential at a point P, which is at a distance from the centre of a small magnet whose "pole strength" is m, has the value

$$V = \frac{m}{r_1} - \frac{m}{r_2} = \frac{m}{r_1 r_2}(r_2 - r_1)$$

where r_1 and r_2 are the distances of the north and south poles respectively, from P. If θ be the angle which the length of the magnet, drawn from south to north pole, makes with the direction of P as seen from the centre of the magnet, the difference of lengths $(r_2 - r_1)$ is practically equal to $2l\cos\theta$, where l is the

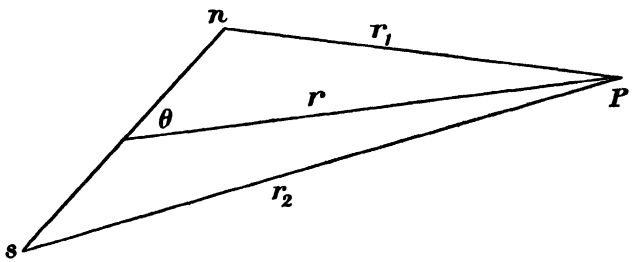

FIG. 2.

semi-distance between the poles. Also r_1 and r_2 may be replaced by r. Hence

$$V = \frac{2ml\cos\theta}{r^2} = \frac{M\cos\theta}{r^2}$$

In the case of an electronic orbit this becomes

$$V = \frac{nea\cos\theta}{r^2}$$

θ being the angle between r and the normal to the electronic orbit. Hence the radial component of the force is

$$F_r = \frac{2nea\cos\theta}{r^3} = \frac{2M\cos\theta}{r^3}$$

and the tangential component is

$$F_t = \frac{nea\sin\theta}{r^3} = \frac{M\sin\theta}{r^3}$$

These expressions are of importance in applications to the mutual actions of the molecular magnets which constitute a magnetized body. If the effective electrons whose motions originate the magnetic quality of the atoms are the most easily detached external electrons, the diameters of whose orbits are of the magnitude of the atomic diameters, the expressions for F_r and F_t cannot strictly be applied to the mutual actions in near atoms. And it is precisely these atoms which contribute most to the internal actions in a solid. On the other hand, if the more internal electrons are chiefly effective, the formulæ are applicable. In this connection it has to be noted that the more internal electrons are the more stable.

Though the proper very complicated expressions for use instead of the above formulæ can be written down, the absence of knowledge regarding phase conditions in neighbouring atoms would compel a process of averaging, which could not lead to an essentially different result. The formulæ apply in the case of fixed positions and orientations of orbits. The postulate that they can be used for all orientations must be regarded as one to be tested by comparison of deduced results with observation. In that respect the conditions to-day are not essentially different from those which held twenty years ago, regarding which, as necessitating the employment of equivalent magnets, Kelvin then wrote: "I believe it represents correctly interesting truth regarding Magnetism in solids; which will be added to when we know the physical quality of a molecular magnet and its relation to ether and to torrents of electrions through ether."

10. Diamagnetism and Paramagnetism.—If molecular magnets be freely suspended, for example by solution of a magnetic salt in a liquid, they will in general tend to turn into alignment with any magnetic field which acts upon them, the north poles pointing in the direction of the field. And the same tendency is manifested even when the molecular magnets are bound by the molecular forces into a solid body. But the hypothesis of Ampère suggests the possibility of the opposite action. For a closed electric circuit, in which no current is flowing, can have a current induced in it by the establishment of a magnetic field which passes through the loop formed by the circuit. Therefore the electrons are acted upon by a force due to the establishment of

the field, which, in accordance with Faraday's law of induction of currents, lasts while the field is being established, and ceases when the field becomes steady.

The inertia of the electrons would maintain their flow, and therefore the flow of the current, without diminution, if no retarding action existed. But the electrons collide or interact with the atoms of the solid conductor, and so interchange with them kinetic energy of corpuscular motion. Thus the originally directed energy of the electrons is rapidly transformed into random kinetic energy of the atoms or molecules, or of the electrons themselves. When this process is completed the current ceases. Hence Ampère's postulate of the origin of the permanent magnetic quality of an atom, as resident in the permanent flow of electricity in atomic circuits, imposes the condition that the electrons are unimpeded in their flow within the atoms. And an interesting support of the legitimacy of his assumption is found in the fact that, at temperatures near the absolute zero, when the atoms are nearly deprived of their translatory motions, a current induced in a copper circuit persists for days before it becomes inappreciable. Now the positive direction of a magnetic field being defined as that in which a north magnetic pole is driven when placed therein; and the rectilineal flow of positive electricity being related to the positive direction of the surrounding circular magnetic field, according to the law of translation and rotation, respectively, of a right-handed screw; it follows that the circular flow of positive electricity in a circuit is also related by the right-handed law to the direction of the resulting magnetic field passing through the circuit. If, therefore, a current of electricity is induced in a circuit by the approach to it of the north pole of a magnet, conservation of energy requires that the field produced by the induced current must oppose that of the magnet. For mechanical work done on the magnet is the only source of the electric energy associated with the induced current. Therefore the electrons must be flowing right-handedly round the circuit as viewed from the approaching magnet.

In the same way, an electron circulating right-handedly, in an atomic orbit in relation to an imposed magnetic field, will have its speed increased by an increase of the field. If it be circulating left-handedly, its speed will be reduced. In the former case the north pole of the equivalent atomic magnet is pointing oppositely

to the imposed field; in the latter, the equivalent magnet is co-directed with the field. In the latter its magnetic moment is decreased, but so long as the moment is not reversed in sign, the imposed field does not tend to reverse the magnet. If reversal of the moment occurs in consequence of a sufficient increase of the applied field, this case reverts to the former. And now the field tends to orientate the atomic magnet from its alignment oppositely to the field. Any deviation from the position of symmetry brings in a magnetic torque. But the revolving electron endows the atom with a moment of inertia, and an axial spin; and accordingly precession ensues.

The action is exactly analogous to that of a top spinning, and precessing stably round a statically unstable direction. The top does not fall over under gravity, and the electronic orbit does not invert itself relatively to the field. It remains orientated reversely, with its magnetic axis pointing on the whole away from the positive direction of the field in a statically unstable orientation. If we consider a body formed of atoms conditioned in this way with regard to the earth's field, we have a body which exhibits north magnetism at its south pointing end. This is the well-known case of a *diamagnetic* body, as distinct from a *paramagnetic* body. Thus, on Ampère's theory, the phenomena of diamagnetism are readily explained, and are continuous with those of paramagnetism.

We say that the currents are flowing paramagnetically or diamagnetically, according as their equivalent magnet is orientated with or against the inducing field. And the body, within whose molecules or atoms these currents are so flowing, is similarly said to be a paramagnetic body or a diamagnetic body.

The stable position of an elongated body with reference to the field is that of greatest exhaustion of potential energy. If the body be paramagnetic, that condition is satisfied when the north and south magnetized ends are separated as far as possible. If it be diamagnetic, the condition is satisfied when the north and south resultant magnetizations are separated as little as possible. Thus an elongated paramagnetic body turns its length parallel to the field, and an elongated diamagnetic body turns its length perpendicular to the field.

A student of Junior Physics is aware of the immense power of simple explanation of apparently recondite phenomena which

lies in the molecular theory. Examples are: gaseous pressure, Boyle's law, surface tension, crystalline form. And a student of Advanced Physics has a still wider acquaintance of its power in various branches, mechanical, optical, thermal, etc. He also knows how the consensus of evidence from observations of a multitude of independent physical phenomena gives us confirmation of estimates of molecular size and molecular number: so that the molecular theory is practically an established fact.

Its applications in Magnetism are unavoidable.

11. Ferromagnetism.—If the electronic currents are sufficiently powerful the induced reverse currents will have little influence, and freely suspended magnetic molecules or atoms will tend towards alignment of their magnetic axes with the imposed field, if it is presumed to act alone on each atomic magnet; that is to say, if it is presumed that the individual atoms or molecules are practically isolated from mutual magnetic influence. In the case of a magnetic liquid, such as liquid oxygen, in contrast with a solution of a magnetic substance in a non-magnetic liquid, we cannot presume that this mutual influence is absent. But, even in that case, the usual temperature conditions are such that the mutual actions are essentially random on the average, and are therefore effectively cancelled. Because of the thermal motions the external field will not succeed in fully aligning the molecular magnet with itself.

These conditions occur in the paramagnetic state. In the case of solids, on the other hand, the molecular forces are sufficiently powerful to maintain the individual atoms on the whole in the same relative positions, which on the small scale at least are crystallinely arranged. If solidification took place with sufficient slowness the entire solid might be uniformly crystalline. The field due to a neighbouring molecule may now, in addition to the thermal effects, tend to prevent complete alignment with the external field; and other peculiarities appear which are characteristic of magnetic quality in crystalline solids. Since they are powerfully exhibited in iron, this state is called the *ferromagnetic* state.

The name is rather unfortunate, for diamagnetic solids are also crystalline, and can exhibit similar peculiarities. The term "magneto-crystalline" state might have been better.

12. Susceptibility: Permeability.—These terms are employed in the same sense as that in which they were employed in the old non-molecular theories. The intensity of magnetization, I, is related to the field intensity, H, by the condition

$$I = kH$$

where k is the susceptibility. And the intensity of induction, B, is related to the field intensity by the similar condition

$$B = \mu H$$

The definition of B as the total flux of force per unit area due to the external field and the magnetization gives

$$B = H + 4\pi I; \quad \mu = 1 + 4\pi k$$

On the old view, I was regarded as a surface density of a continuous distribution of magnetism. The values of the quantities are determined externally to the magnetized body at distances which are usually very large in comparison with molecular distances. In this respect there is no difference between the old and the modern results. The difference appears in the condition that, the molecular theory being postulated, it has to predict numerical values at external points when the molecular or atomic data are given. And this statement extends to the prediction of directional peculiarities in the case of magneto-crystallic phenomena. The old theories imported directional quality of this type by introducing three independent susceptibilities in three independent directions. They gave a general correspondence of theory to observation in magneto-crystallic action; but the laws deduced do not agree in detail with those which follow on the molecular basis.

The molecular value of the susceptibility and magnetization become important in the molecular treatment. The magnetization I, being the quantity of magnetism per unit area, is equal to the magnetic moment per unit volume; and this is equal to the maximum torque per unit volume, in unit field, if the portion of the magnetized body of unit volume were isolated. It is also equal to the work contributed per unit volume by unit field in

the establishment of the magnetization. *The specific magnetization* (σ) is the magnetization *per unit mass*, and is expressed by

$$\sigma\rho = I$$

where ρ is the density. It is the *magnetic moment per unit mass.*

The ratio $\sigma/H = \chi = k/\rho$ is called the *coefficient of magnetization.* It is the susceptibility per unit of mass. So if we multiply it by the molecular weight we get the *molecular susceptibility*, that is, the susceptibility χ_m per gramme-molecule.

From this again we find the important *magnetic moment per molecule*

$$\chi_A = \frac{\chi_m}{A} = \frac{k}{n}$$

where A is Avogadro's number, $60{\cdot}6\ (10)^{23}$, and n is the number of molecules per unit volume.

CHAPTER II

THE FOUNDATION OF THE MOLECULAR THEORY OF MAGNETISM AND DIAMAGNETISM

13. The Non-Molecular Theory.—The molecular constitution of material bodies is not directly evident to the senses. Therefore, in the case of phenomena which are directly evident, it is more convenient to give descriptions and explanations in terms of ideas of continuous structure, and it is natural that the ideas imported into discussions of the more recondite phenomena should be based on our knowledge of large-scale appearances; in particular, that heat, and electricity, and magnetism, should be presumed to be fluids, though subject to restrictions such as the absence of gravitational effects.

In the main that treatment can lead to results which are in good agreement with observation, and also with the molecular treatment. Thus the pole-face of a magnet, uniformly magnetized in a direction normal to the face, is usefully regarded as exhibiting a continuous surface distribution of "magnetism" of definite intensity per unit area, while, on the molecular theory, that "intensity" becomes the sum of the magnetic moments of all the molecules included in unit volume of the material. The outside field is *practically* continuous since we cannot test directly for the existence of discontinuity.

But indirect evidence is sometimes available especially when isotropy is absent. Even when it is present, special features may point to a molecular constitution. These do not exist, for example, in the case of conductors of electricity; but induction in dielectrics at once raises the molecular aspect; and, in magnetism, it is directly raised by the absence of conduction. Even in these cases, the postulate of continuous action is usefully maintained by the introduction of the ideas of specific inductive capacity and of permeability, and is extended into the region of non-isotropy by the introduction of the ideas of three different principal capacities

or permeabilities. Here doubt may properly arise since the molecular quality is polar. Thus, in the case of a cubic crystal, the magnetic properties are not isotropic; and the question is directly raised whether, or not, the non-isotropy deducible from the hypotheses of three different permeabilities leads to the same result as the complete molecular treatment leads to.

Since a cubic crystal is not magnetically isotropic, the three principal permeabilities must be referred, not to the direction of the three sets of edges, but to the directions of the edges, the face diagonals, and the body diagonals. And it is an assumption, though the simplest, to regard these three component permeabilities as capable of preserving constant values for all directions of magnetization. Actually this condition does not subsist, though it can be used as a first approximation.

14. Weber's Foundation of the Molecular Theory.—As Gauss pointed out, the application of the non-molecular theory enables us to determine uniquely a distribution of magnetic fluid over the surface of a magnet which will produce the same external field as the magnet itself produces, and so can *avoid* any consideration of the internal conditions of the magnet while giving a complete practical account of its external action. But, if the magnet is, say, a crystalline sphere, although the suitable surface distribution can be found for any direction of uniform magnetization of the sphere, it still remains to explain the variation of the distribution with the direction of magnetization; and that problem cannot be attacked until the internal constitution of the magnet is specified.

These conditions, apart from special reference to crystalline action but nevertheless including it, are clearly pointed out by W. Weber in his classical and epoch-making paper, **On the Connection of the Theory of Diamagnetism with the Theories of Magnetism and Electricity,** published in 1852. He states that we discriminate permanent and transient magnets, of which hard steel and soft iron magnets, respectively, are examples; but the magnetism of permanent magnets can be investigated through their actions, while, in addition to the action, the origin of the actions has to be taken into consideration in the case of transient magnets. He points out that similarly, diamagnets might be classified as permanent and transient, but

there would then be nothing to distinguish permanent diamagnets and permanent magnets. So that only transient diamagnets come into consideration, and that in two ways, partly in regard to their origin, partly in regard to their actions. He says that no conclusion regarding the true internal state can be attained from consideration of the actions alone. Another consideration, resting on a basis entirely outside the actions, must be introduced.

He says that all possible causes of diamagnetism, like those of magnetism, are classifiable into external and internal. The external cause is given by observation, as the actions themselves are. It is the same for magnetism and diamagnetism—a definite magnetic or electromagnetic field of force. If, in addition to this external field, the internal field lying within the body itself were known, the diamagnetism itself would be completely defined; and conversely, a way is opened up for the estimation of the unknown internal cause, if, besides the known external origin, the diamagnetism resulting from both is known. If one connects the known magnetic field with the ideal distribution investigated from the actions, whether for iron or bismuth, it follows that identical fields originate *opposite* distributions in iron and bismuth; or conversely, that an identical distribution in iron and bismuth corresponds to oppositely directed fields. The reason why contrary external origins produce the same effects in iron and bismuth must lie in the difference of the internal causes in iron and bismuth. In order to investigate more closely the difference, shown in this way, between the internal origins in iron and bismuth, it is necessary to *classify* all possible internal origins, which can have actions explainable in terms of one ideal distribution; and then to test whether, under all that we can enumerate, such are included as give account of the just mentioned antithesis, actually present in magnetic and diamagnetic bodies under the same external influence.

15. Weber's Classification of the Internal Sources.—Weber points out that we can state four essentially different kinds of internal sources, seated in bodies, which are capable of originating actions explainable by an ideal distribution of magnetic fluids.

(1) The internal source of such actions can be contained in the existence of two magnetic fluids, which are more or less movable independently of their ponderable carrier.

(2) It can be contained in the existence of two magnetic fluids which are *movable only with the molecules of their ponderable carriers* (rotatable molecular magnets).

(3) It can be contained in the existence of *steady molecular currents, formed of the electric fluids*, which are rotatable *with the molecule*.

(4) It can be contained in the existence of the electric fluids which can be set in *molecular streaming motion*.

He remarks that "these are the only causes of the requisite nature which one knows and can subject to test; that the first case lies at the base of the theory of magnetism developed by Coulomb and Poisson: the third case lies at the base of the relation, developed by Ampère, between the theory of magnetism and electrodynamics; the second case is reducible to the third in consequence of the theorem proved by Ampère, that molecular magnets and molecular currents are equal in all their actions when substituted for each other; there remains thus only the fourth case, previously unconsidered and uninvestigated.

"This fourth case," he says, "postulates electric fluids which can be set in molecular streaming; but the possibility of being set in molecular streaming rests for the electric fluids on this, that within the single molecules, or surrounding them, re-entrant paths are present in which these fluids are movable without resistance, wherefrom it follows that then only an electromotive force (i.e. a force which acts in opposite directions on the positive and negative fluids) is required, in the direction of this path, in order to actually move the fluids in that path. But the theory of magneto-electricity shows that, through the increase or diminution of a magnetic field, an electromotive force is actually produced which acts in opposite directions upon the two electric fluids and must set them in streaming motion. The direction of this molecular current is given in its dependence on the increase or decrease of the magnetic field, by the fundamental law of magnetic induction, and again the ideal distribution is dependent on this molecular current according to the relation, developed by Ampère for the third case, of electro-dynamics and the theory of magnetism. We obtain, as directly also, the relation between the ideal distribution and the increase or diminution of the magnetic field to which it corresponds.

"But it is evident from this," he proceeds, "that at the moment

at which the increase or diminution of the magnetic field occurs, such a molecular current must be produced, and that these successively produced currents, if they do not again vanish, must be additive.

"But these currents do not themselves vanish; for Ampère has shown that permanence must be ascribed to the molecular currents, i.e. the electric fluids suffer, in their circulation round the ponderable molecule, no such resistance as the electric fluids which flow in a ponderable conductor suffer, by which the rapid diminution of the electric currents is explained. (It is evident, from this permanence necessarily characterizing the molecular currents, that the possibility of setting the electric fluids in molecular streaming, as has been already mentioned, rests upon this, that in the molecules, or around them, re-entrant paths are present in which these fluids move without resistance.) Therefrom it follows that with stronger increase of the magnetic field, a stronger accumulation of the magnetic fluids in the ideal distribution is bound up, wherefrom it follows that, to each given strength of the magnetic field, there corresponds a definite moment of the ideal distribution. But this summation only occurs in the molecular currents, because only in them does the motion of the electric fluids experience no resistance. The other currents which are produced by the given field in wider circuits, but which quickly vanish through the resistance which they find in the circuits, produce only at the moment of their excitation (by increasing or diminishing field) magnetic actions on other bodies, which vanish at once as soon as the field becomes constant, and which therefore stand in no relation at all to the actually present field, which relation nevertheless must necessarily exist if account is to be taken of variable magnets or diamagnets, for which therefore only molecular currents are suitable. If now, with regard to these molecular currents one develops, in accordance with the law of magnetic induction, the dependence of the moment of the ideal distribution upon the magnitude of the actually present field, one finds that, if in the line of the magnetic force that direction is denoted as positive wherein the north pole of a magnetic needle is driven, and if one determines, for the ideal distribution dependent on this field, the centres of inertia of the north and south fluids, the former of these two points is, relatively to the latter, driven in the *negative direction*, i.e. *exactly*

inversely as for the three other cases, whence conclusion can be reached regarding the internal source of diamagnetism."

16. Weber's Discussion of the Source of Diamagnetism.—"This remarkable result," he proceeds, "permits thus an application to the foundation of a *Theory of Diamagnetic Phenomena*, which takes account of the forces producing it, respecting which theory has hitherto failed. It is not sufficient for such a theory that one can suitably represent the Diamagnetic condition of a body, relatively to all its actions, by an ideal distribution of magnetic fluids over its surface. But it is essentially requisite thereto that also account be taken of the forces by which the condition is produced, as also according to what laws, and upon what, these forces act.

"From the above synopsis and consideration of the different possible ways in which the condition of a body representable by an ideal distribution of magnetic fluids can originate, only one single case has been found in which a law, agreeing with the fundamental phenomena of the origination of diamagnetism, results for the dependence of the ideal distribution of the magnetic field. It follows that, regarding the origin of the diamagnetic condition of bodies, account can only be given if this case is regarded as actually occurring, according to which the increase of diamagnetism of a body is proportional to the inducing force which acts upon the electric fluids, present in the body, movable in definite circuits, without resistance, round the body, and accelerates the speed of their motion in these paths. Thus, for example, the diamagnetism of bismuth is explained by the assumption that the bismuth molecule *contains in itself re-entrant paths (or canals) in which the electric fluids are movable without resistance*, whilst these fluids are only movable in all other paths by the overcoming of a resistance proportional to their speed. The occurrence of a pure diamagnetism (not mingled with magnetism) requires in addition that the molecules containing these paths or canals are *not rotatable*, because otherwise *rotatable molecular currents* could occur which would be so strong that a part of their intensity could, by the rotation, be regarded as constant, and would have as consequence, according to Ampère, a magnetic condition. Conformably to this assumption, the diamagnetism or electro-diamagnetism of a body can be

determined from the magnetic or electromagnetic field acting upon it."

17. Weber's Determination of Diamagnetic Moment.— From Faraday's law of electromagnetic induction we know that if a magnetic field of intensity H be suddenly made to pass normally through the interior of a circular conducting circuit of radius r, the time integral of the total electromotive force which is induced round the circuit is

$$-\pi r^2 H$$

Consequently its amount per unit length is

$$-\frac{rH}{2}$$

This is the total force acting on unit quantity of electricity supposed to be distributed over unit length of the circular circuit. So, if μ be the mass of unit quantity, the speed developed in each unit of quantity is, by the second law of motion,

$$-\frac{rH}{2\mu}.$$

Therefore, if the quantity of electricity per unit length is ϵ, the corresponding induced current is

$$-\frac{rH\epsilon}{2\mu}$$

Consequently (§ 8) the diamagnetic moment is

$$-\frac{\pi r^3 H\epsilon}{2\mu}$$

Now consider all the circuits per unit mass of the diamagnetic substance. These are randomly arranged on the average. If the normal to one of these be inclined at an angle θ to H, the above expression has to be multiplied by $\cos\theta$. But, on account of the randomness of the distribution, corresponding to every such circuit there is another anti-symmetrically inclined at the same angle. Therefore the resultant moment for that pair, and conse-

quently for every such pair, is along the line of the field. So we have finally to multiply again by $\cos\theta$. Thus the diamagnetic moment is the sum of all the quantities

$$-\frac{\pi r^3 H\epsilon}{2\mu}\cos^2\theta$$

If there are n circuits per unit mass of the material, the directions of their normals may be specified by n points scattered randomly over the surface of a unit sphere. The number of these per unit

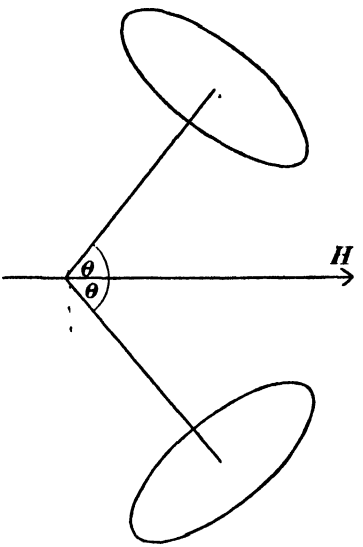

FIG. 3.

area of the surface is $\nu = n/4\pi$. So the number of normals which are inclined to H at angles lying between θ and $\theta + d\theta$ is $\nu \cdot 2\pi \sin\theta d\theta$. Therefore the total diamagnetic moment per unit mass is

$$-\frac{\pi r^3 H\epsilon n}{2\mu}\int_0^{\pi/2}\cos^2\theta \sin\theta d\theta = -\frac{\pi r^3 H\epsilon n}{6\mu}$$

The factor $n/3$ can be found very directly from the consideration that, on the average, one-third of the number of randomly arranged orbits face in one given direction.

Thus the theory shows that the diamagnetic moment of any mass of a substance which exhibits diamagnetic properties is proportional to its mass and to the strength of the magnetic field which is acting upon it. *These are characteristic features of Diamagnetism.*

18. Modification for Electronic Orbits.—The sign of summation must be prefixed to the above expression for the diamagnetic moment if the molecules contain different types of circuits with various values of r or n. Weber's treatment is molecular in so far as the matter is concerned. He employs the Ampèrean unresisting circuits, which are supposed to exist in connection with the molecules. On the other hand, the treatment with respect to electricity retained the early denotation of electricity as a continuous fluid. Since the Ampèrean currents are now known to consist in the circulation of electrons in atomic or molecular orbits, the complete modernization of Weber's theory involves the corresponding modification of his treatment just given. This, as already remarked, is the only point in respect to which Weber's treatment requires fundamental change in order to bring the theory into line with the remainder of molecular theory, and also for the specific explanation of certain phenomena (§§ 20, 21, 33).

The quantities e and m being the charge and mass of an electron revolving in a circle of radius r with speed v, the effective current flowing is

$$i = \frac{ev}{2\pi r}$$

for the charge e passes any given point of the path $v/2\pi r$ times per second. Therefore the equivalent magnetic moment is

$$M = i\pi r^2 = \frac{evr}{2}$$

But the total electromotive force induced in the circuit is the integral of

$$E = -\pi r^2 \frac{\partial H}{\partial t}$$

where H is the intensity of the inducing magnetic field taken normal to the plane of the circuit.

Hence the strength of the electric field acting on e is

$$\frac{\pi r^2 \frac{\partial H}{\partial t}}{2\pi r} = \frac{r}{2}\frac{\partial H}{\partial t}$$

and the time integral of the force acting on e during the establishment of H is

$$\frac{rHe}{2}$$

This is equal to the change of momentum mu produced in the electron, u being the diminution of speed. So we have

$$u = \frac{reH}{2m}$$

and the diminution in the current strength is

$$\frac{eu}{2\pi r} = \frac{e^2 H}{4\pi m}$$

Thus the change of the magnetic moment is

$$\mu = -\frac{eu}{2\pi r} \cdot \pi r^2 = -\frac{e^2 r^2 H}{4m}$$

As in § 17, to take account of all orbits randomly arranged in unit mass, we must multiply by one-third of n.

So the induced diamagnetic moment, per unit mass, is

$$-\frac{e^2 r^2 H n}{12m}$$

This result may be deduced directly from the expression

$$-\frac{\pi r^3 H \epsilon n}{6\mu}$$

of § 17. For, μ being the mass of unit quantity, we get

$$\mu e = m$$

And also, if e and m be regarded as uniformly distributed round the circumference, we have

$$2\mu r\epsilon = e$$

19. Electronic Orbits (continued).—In the treatment just given it has been tacitly assumed that r is unaltered by the imposition of H. This requires justification unless the postulated sub-atomic conditions are to have a strongly *ad hoc* nature.

The centrifugal force, if r does not alter, becomes

$$\frac{m(v-u)^2}{r}$$

So its decrease, if r is to be unaltered, must be balanced by the ponderomotive force, acting on the electron because of its circulation in a magnetic field. Thus

$$\frac{mv^2}{r} - \frac{m(v-u)^2}{r} = He(v-u)$$

$$2mvu - mu^2 = Her(v-u)$$

or

$$mu\left(2 - \frac{u}{v}\right) = Her\left(1 - \frac{u}{v}\right)$$

But, if $2mu = Her$, the ratio $u/v = e^2r^2H/4mM$ must be negligible. With the values $e = 1\cdot57(10)^{-20}$, $e/m = 1\cdot77(10)^7$, $r = 2(10)^{-8}$, $M = 2(10)^{-18}$, we find that u/v is of the order $(10)^{-11}H$. So that, even with $H = 10^5$, u is only about one millionth part of v.

20. Electronic Orbits, Gyrostatic Moment.—The induced currents which flow in the Ampèrean paths, in consequence of variation of an external magnetic field, are postulated to obey Faraday's laws, which are established for ordinary metallic conducting circuits. But it is now known that these currents are really convection currents due to the transference of free electrons, contained in the metal, under the action of the induced electromotive force. So it is quite reasonable to postulate that the electromotive forces act in the same way on electrons whose paths are situated within the molecules or atoms themselves

instead of being free. Yet the fact that it is a postulate to assert that they do so is clear when we consider that it is within the atom that we have evidence of departure from the classical laws of mechanics. For, if dynamical actions are at root electrodynamical we cannot then be certain that the latter do not fail in atomic applications.

Apart from the action of electromagnetic induction, there is another source of diamagnetic moment when a magnetic field acts on a diamagnetic body. This arises from the ponderomotive force which acts on a conductor carrying a current in a magnetic field. In the case of the convection current carried by a revolving electron, the force acts upon the electron which carries the charge. In a fixed magnetic field parallel to the orbit, the force is in one direction during one half of the revolution, and is in the opposite direction during the other half. Thus it acts in the same way as a mechanical torque, which causes the orbit to precess. And this precession gives rise to the diamagnetic action.

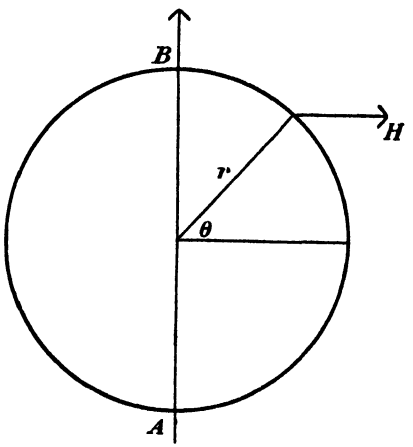

FIG. 4.

No torque comes into play if the field is perpendicular to the plane of the orbit; for the mechanical force then lies in that plane. It is that component, of any field, which lies in the plane of the orbit that causes the torque.

Suppose that the orbit is circular of radius r, and consider a radius vector which makes an angle θ with the direction of the field H. The effective current i due to the motion of the electron is $ev/2\pi r$. If the electron circulates right-handedly the direction of the force acting on the element $rd\theta$ is perpendicular to the diagram, and is inwards or outwards from it according as θ is in the first or second quadrant, or in the fourth or third, respectively.

The torque is therefore round the line, in the plane of the

diagram, which is perpendicular to H. And its positive direction is indicated by the arrow. The force acting on the electron while it describes the element $rd\theta$ is

$$H \cos\theta \,.\, evrd\theta/2\pi r$$

and its moment round AB is found by multiplying the force by $r \cos\theta$. Thus the moment is

$$\frac{Hevr}{2\pi} \cos^2\theta d\theta$$

Multiplying by 4 and integrating from $\theta = 0$ to $\theta = \pi/2$, we find that the total torque is

$$T = \frac{Hevr}{2}$$

21. Gyrostatic Moment (continued).—Now the moment of inertia of the electron is mr^2. And, by the second law of motion, the torque is equal to the rate of change of angular momentum, that is of $mr^2\omega$. But the torque is round AB, and the angular velocity ω is round the central axis, perpendicular to the diagram, in the direction through it from the front. So the axis of torque, that is of angular acceleration, being at right angles to the axis of spin, the torque cannot alter the rate of spin. Thus the angular velocity v/r remains constant, and the angular momentum round the axis of torque, AB, can only increase under the torque by precession which tends to bring in, round the axis of torque, a component of the spin. So the system rotates, round the central line parallel to H, right-handedly round the direction of H. And it does this with a constant angular velocity of precession Ω, say. For v remains constant, and H always lies in the plane of the circle. Thus the torque, being equal to the rate of change of the angular momentum, is equal to the constant angular momentum multiplied by Ω. So

$$T = \frac{Hevr}{2} = mr^2 \cdot \frac{v}{r} \cdot \Omega$$

And we get

$$\Omega = \frac{He}{2m}.$$

This constant precessional spin, which is equal to the angular velocity u/r (§ 18) caused by the Faraday effect, carries negative electricity right-handedly round the direction of H. It therefore gives rise to a diamagnetic moment. To calculate the magnitude of this, consider the element $rd\theta$. It moves in a circle of radius $r\sin\theta$ with angular velocity Ω and carries a negative charge $erd\theta/2\pi r$. So the equivalent current is

$$i = \frac{ed\theta}{2\pi} \cdot \frac{\Omega}{2\pi} = \frac{He^2 d\theta}{8\pi^2 m}$$

The corresponding diamagnetic moment is

$$ia = -\frac{He^2 r^2}{8\pi m} \sin^2\theta d\theta$$

By integration, the total moment for the circuit becomes

$$-\frac{He^2 r^2}{8m}$$

If H were inclined at an angle ϕ to the plane of the circle we have to replace it by $H\cos\phi$. And, since $+\phi$ and $-\phi$ occur in pairs, symmetrically, in the case of a random distribution of n circuits per unit mass, the resultant is in the line of H, so that we must resolve again by $\cos\phi$. Thus the number orientated between ϕ and $\phi + d\phi$ is

$$\frac{n2\pi \sin\phi d\phi}{2\pi}$$

Therefore the resultant diamagnetic moment is

$$-\frac{He^2 r^2}{8m} \cdot 2n \int_0^{\pi/2} \sin\phi \cos^2\phi d\phi = -\frac{He^2 r^2}{12m} n$$

This is the same value as we found in § 18 for the resultant moment as produced by the component of H normal to the circuit. So the total diamagnetic moment, if both actions occur, is

$$M = -\frac{He^2 r^2}{6m} n$$

22. Steady Ampèrean Currents. Gyrostatic Effects.—The diamagnetic effect due to the electromotive force induced by the imposition of an external magnetic field has been discussed in § 17. In the same way we might investigate the effect due to the component of the field parallel to the atomic circuit; but it can readily be deduced from the expression just found. The whole electronic charge e, and its whole mass m, have to be replaced by the charge per unit length, and the mass per unit length. The value of the ratio is obviously unaffected. But e, as the total quantity, has to be replaced by ϵ as the amount per unit length multiplied by $2\pi r$. The expression for the total *gyrostatic* moment per unit mass therefore becomes

$$M = -\frac{\pi H \epsilon r^3 n}{6\mu}$$

which is equal to the effect deduced from the induced currents in § 17.

23. Condition for Manifestation of Diamagnetism.—Even in bismuth, which is the most strongly diamagnetic body known, the specific diamagnetic moment developed by the application of a given magnetic field is feeble in comparison with the magnetic moment which the application of the same field would produce, in any of the more pronounced magnetic substances. That is to say, the induced molecular currents are feeble in comparison with the permanent molecular currents, whose existence is postulated in the case of magnetic bodies.

Consequently, the diamagnetic condition arising from induced electromotive force acting in the molecular circuit may be present in a magnetic body though only to an extent which is negligible in comparison with the pre-existing magnetic condition. Then, in order that a substance should be classifiable as diamagnetic, it is essential that either no permanent molecular currents occur, or they must occur in balanced groups.

24. Weber's Introduction of Mutual Molecular Action.—Weber points out that, in his discussion of the magnitude of the diamagnetic moment (§ 17), the result has been deduced on the assumption that each molecule could be treated by itself. He proceeds to say that "This is strictly speaking not the case,

but in addition, those electromotive forces are co-active which arise from the *mutual action* of the diamagnetic molecules, just as not only does the external field, say the earth's field, act upon a particle of an iron rod, but also those fields which arise from the mutual action of the magnetic particles of the iron upon one another."

Here we have the explicit statement which completes the foundation of the modern theory of molecular magnetism.

Weber then remarks that when consideration is made of this mutual action of diamagnetic molecules, although it is so small as to be almost negligible, it nevertheless brings into view a noteworthy discordance which exists between the mutual actions of diamagnetic and magnetic molecules.

25. Weber's Application of Mutual Diamagnetic and Magnetic Actions.—Weber next applies the theory to the

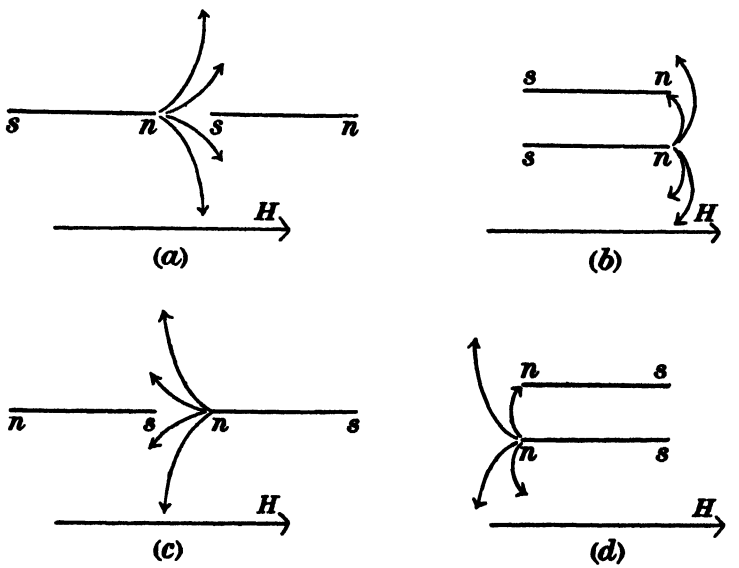

FIG. 5.

investigation of the mutual actions. He limits his investigation to the case of two particles, supposed to be small in comparison with their distance apart; and he, further, restricts it to the "end-on" and "broadside-on" position. In spite of the magnitude

of these restrictions, he sees clearly that this theory will lead to a powerful experimental discrimination between magnetic and diamagnetic substances, well known to exist.

If two magnetic particles be placed in the end-on position, the axis and line of centres being coincident in direction with the external field H, the field acting on one in virtue of the presence of the other is co-directed with H. In the broadside-on position it is contrariwise directed [Fig. 5 (*a*) and (*b*)].

In the same position, but with diamagnetic particles in equilibrium with the external field [Fig. 5 (*c*) and (*d*)], the mutual actions are contrariwise and co-directed, respectively.

If the magnetic moment of the particles have the numerical value M, while the distance between centres is r, the total field is

$$H \pm \frac{2M}{r^3}$$

respectively. Thus the magnetic and diamagnetic moments developed are respectively

$$\pm M\left(1 \pm \frac{2M}{Hr^3}\right)$$

in the end-on position. Similarly they are

$$\pm M\left(1 \mp \frac{2M}{Hr^3}\right)$$

in the broadside-on position.

Weber therefore concludes that, whilst one, in order to develop the strongest magnetism in a mass of iron in a given field, uses it as a long thin rod orientated parallel to the field, on the contrary, to develop the strongest diamagnetism in a mass of bismuth, one must use it in the form of the thinnest plate placed perpendicularly to the field (see p. 12, foot).

His approximate formulæ, as given above, exhibit the first instance of the application of the molecular theory to the determination of the nature and magnitude of the internal field due to the mutual action of molecular magnets. The brief investigation shows his clear recognition of the essential definiteness of the problem of its evaluation.

26. Dependence of Magnetization on the Magnitude of the Magnetic Field.—Weber points out next that, just as the molecular theory of diamagnetism leads to the conclusion that the diamagnetic moment, developed in a magnetic field, is proportional to the field strength, so, on the older view of magnetic fluids movable in the iron molecules, the moment developed in the iron should be proportional to the field strength. But, he adds, we must now abandon this view along with that of the magnetic fluids in the interior of bodies, and, instead, adopt Ampère's view that the iron molecules are the rotatable carriers of permanent molecular currents, from which *a different law of the dependence of variable magnetization on the field strength follows.*

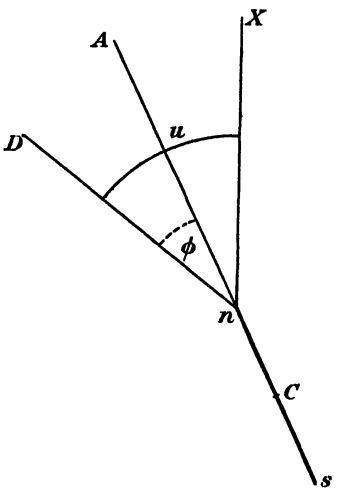

FIG. 6.

At that time the experimental data regarding the phenomena of magnetization were very limited, so that a full development of the theory on Weber's lines, as described at the end of the preceding section, was largely uncalled for. In the first place he expressly limited his investigation to the case of ideally soft iron, that is, to substances which do not exhibit residual magnetization. And in the next place, instead of giving a detailed evaluation of the internal field, he substituted for it, in the usual way followed when the actual form of a law is unknown, the simplest postulated expression which would render a good account of the phenomena contemplated.

His statement (see Fig. 6) was the following: Let *ns* be the axis of a permanent molecular current which can turn round its centre *C*. Let its direction, when in equilibrium, under the action of zero external field $X = 0$, be parallel to *nD*. The fact that, in soft iron, the magnetism developed by a field acting on the iron again vanishes whenever the field is withdrawn, proves that the molecular current from the rotation of which the magnetism arises, is again drawn back into its original orientation

parallel to nD. But, he adds, this reverse force "originating in the mutual action of the molecules" must increase with the deflection and may be represented by

$$D \sin \phi$$

where D denotes a constant which may be called the molecular directive force.

The postulate here made is that D is constant *in magnitude and direction* as ϕ increases. A complete development of Weber's "mutual action of the molecules" shows that the postulate is incorrect. Yet it was sufficient in Weber's hands to give a good general account of the phenomena of magnetization of soft iron.

When the external force X acts in the direction nX which makes an angle $XnD = u$ with the direction of the internal field, the molecular current is deflected through the angle $AnD = \phi$, and the new position of equilibrium is given by the equation

$$X \sin u \cos \phi = (D + X \cos u) \sin \phi$$

or

$$\tan \phi = \frac{X \sin u}{D + X \cos u}$$

From this deflection the increase of the magnetic moment of the molecular current, in the direction of the force X, can be estimated. The quite invariable magnetic moment of the molecular current being μ, the moments in the direction of the force before and after deflection were $\mu \cos u$ and $\mu \cos (u - \phi)$ respectively. So the increase desired is

$$x = \mu (\cos (u - \phi) - \cos u)$$

Hence, by use of the expression for $\tan \phi$, we get

$$x = \mu \left[\frac{X + D \cos u}{\sqrt{X^2 + D^2 + 2XD \cos u}} - \cos u \right]$$

The molecular currents being on the whole randomly directed, and being n in number, the total moment is

$$y = \frac{n}{2} \int_0^\pi x \sin u du$$

The integration being carried out, we obtain the result,

$$y = \frac{\mu n}{12}\frac{R}{X^2 D}(R^2 + 3X^2 - 3D^2) + \text{const.}$$

as given by Maxwell, where R is the resultant of X and D at the limits of integration. These are $X + D$ at the upper limit, and $\pm (D - X)$ at the lower limit according as $D \gtrless X$.

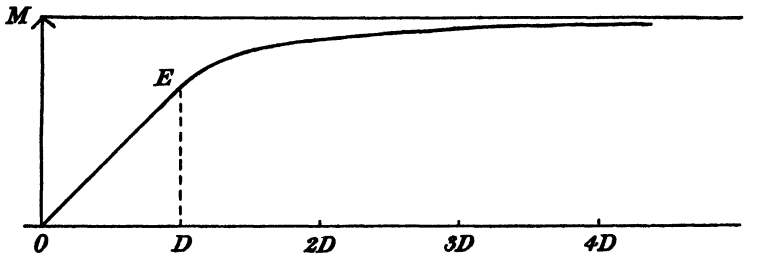

FIG. 7.

The value of y below $X = D$ is $2\mu nx/3D$. At $X = D$, it is $2\mu n/3$; and at $X > D$, it is $\mu n(3X^2 - D^2)/3X^2$, μn being the saturation value. The curve gives a fair general representation of the trend of magnetization of soft iron with increase of field.

27. Maxwell's Introduction of Residual Effects.— Weber's treatment being explicitly limited to the case in which no residual effects occur, Maxwell remarks (*Electricity and Magnetism*, Vol. II) that he "thought it desirable to examine the results of making a further assumption relating to the conditions under which the position of equilibrium of a molecule may be permanently altered." This further assumption was that the axis of a deflected molecule will return to its original position if the deflection does not exceed a certain value θ_0; while, if θ_0 be exceeded, a permanent deflection equal to the excess over θ_0 remains. Concerning it, Maxwell remarks that "this assumption with respect to the law of molecular deflection is not to be regarded as founded on any exact knowledge of the intimate structure of bodies, but is adopted, in our ignorance of the true state of the case, as an assistance to the imagination in following out this speculation suggested by Weber."

This statement is of great importance. Maxwell as well as

Weber knew that the mutual action of the molecules depended on the distribution and orientation of the molecule in the bodies, i.e. on the intimate structure of bodies. Weber selected the case of a random configuration, and so avoided the consideration of positions of equilibrium. Maxwell, wishing to discuss residual magnetization obviously dependent on equilibrium conditions, introduced a simple law for the establishment of new equilibrium positions in the course of magnetization. Mathematically, both, avoiding the definite elaboration of the actual law of force and of the actual conditions of equilibrium, made simple postulates

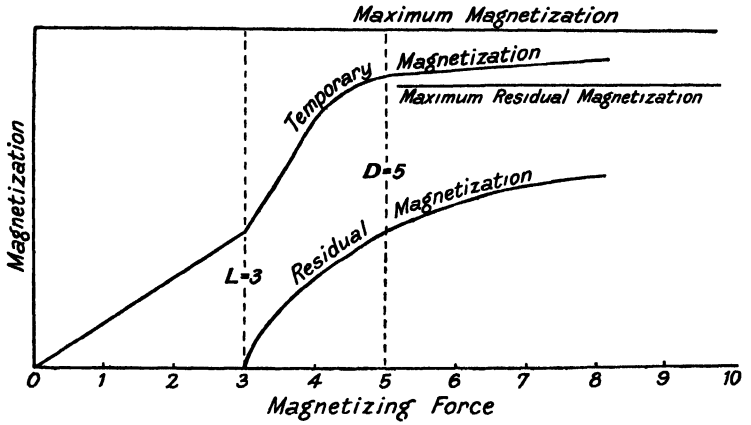

FIG. 8.

regarding each, in order to test if, and how far, a general explanation of magnetic phenomena might be reached on these lines. Replacement of their selected laws by others would only in general lead to changes in details and not to changes in the nature of the result.

Maxwell succeeded in elucidating the characteristic form of the magnetization curve, an initial part of low slope, an intermediate part of high slope, and a final part of lessening slope tending to zero.

And he shows how, by postulation of several values for D and θ_0, a better simulation of the normally observed curves could be obtained.

CHAPTER III

MOLAR EFFECTS OF MOLECULAR MAGNETISM

28. Interaction of Adjacent Molecules.—In the preceding chapter the existence of interaction amongst magnetic molecules or atoms has been briefly discussed in regard to its general nature. When we proceed to more specific details assumptions have to be made concerning the law of interaction.

At molecular distance we cannot, *except for purposes of testing the results of postulates*, assume that the law of the inverse square is applicable. And if for these purposes we assume that it holds, the limitation of the investigation of § 8 to points which are far distant in relation to molecular dimensions may not be justifiable in the case of closely packed molecules. Electronic orbital dimensions seem to be comparable with the mutual distance of adjacent molecules in a closely packed set. So the random placing of an ideal magnet of equivalent moment anywhere within an electronic orbit is unallowable. The evaluation of the internal field is correspondingly complicated.

But if the molecules are randomly orientated throughout a body, certain effects of the imposition of an external magnetic field, which are independent of mutual magnetic interaction amongst the molecules, can be readily deduced.

29. The Gyromagnetic Effect and its Converse.—The possible existence of this molar effect, arising from the molecular magnetic actions, was first pointed out by Richardson. We have found, in the preceding chapter, that circulatory and precessional motions of electrons take place, according to theory, when a molecule is subjected to the action of a magnetic field. Consequently a moment of momentum arises round the direction of the field, in addition to the diamagnetic or magnetic moment which has been discussed in §§ 18–21. If this moment of momentum is communicable, in whole or in part, in conse-

quence of collisions, to neighbouring molecules, a corresponding rotation of the total molecular system should be demonstrable. Conversely, rotation of an unmagnetized body should give rise to magnetization. For by that rotation, circulatory motion of the constituent electrons is established. And that implies, as we have seen, the production of a diamagnetic field in the direction opposite to that of rotation.

In fact, this effect furnishes an illustration of the principle of stability in accordance with which, if an action $+ A$ causes an effect $+ B$, then $+ B$ as a cause, will originate a *diminution* of A as its effect, if the two actions are directly connected. For if an increase of A resulted, the actions would mutually intensify each other, if not without limit, at least to an extent in opposition to the conservation of energy. Thus the institution of a magnetic field gives rise to right-handed circulation of the electrons round the positive direction of the field. Therefore right-handed rotation of the system of molecules constituting a body gives rise to a negatively directed field.

The observed effects are consistent with the predictions in so far as direction is concerned; but the observed magnitudes are inconsistent with them, so that more detailed investigation is needed with respect to the underlying postulates.

30. The Orbital Moment of Momentum.—In the case of paramagnetism the molecular magnetic moment is large in relation to the induced diamagnetic moment. Therefore the paramagnetic moment of momentum of the electron is large in relation to the induced moment of momentum. That is to say, the speed v of the electron is not much altered by the induced speed, which changes it to $v \pm u$, where (§ 18)

$$u = \frac{reH}{2m}$$

if H is normal to the orbit.

In so far as v is concerned, the orbits being initially oriented at random, there is no resultant initial moment of momentum round H. And, H being presumed to be strong enough to give practical saturation, the normals are finally parallel to H. Thus the final paramagnetic moment of momentum is, per unit volume,

$$M = - nvrm$$

where n is the number of orbits of radius r per unit volume. This is so apart at least from the slight change due to the induced speed.

At the same time the magnetic moment, per unit volume, initially zero also, has become (§ 18)

$$G = n \cdot \frac{ev}{2\pi r} \cdot \pi r^2 = \tfrac{1}{2} nevr$$

In this case of paramagnetism, therefore, the ratio of the mechanical moment to the magnetic moment is

$$\frac{M}{G} = -\frac{2m}{e}$$

The negative sign is due to the fact that the electrons are, in the magnetized condition, circulating left-handedly in relation to the direction of H.

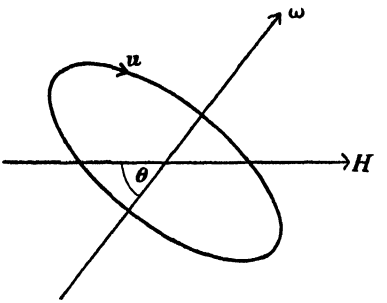

FIG. 9.

Though the term involving u is unimportant in paramagnetism, it is all-important in the case of diamagnetism, in which there is no resultant initial electronic circulation in the molecule. To evaluate the mechanical moment in this case, consider an orbit in which the direction of the angular velocity is inclined to that of H at an angle θ (Fig. 9). Here we have

$$u = \frac{reH \cos\theta}{2m}$$

and the component of the mechanical moment parallel to H is $mru \cos\theta$. Now the number of orbits whose normals are inclined to H within the range $d\theta$ is $n \sin\theta d\theta/2$ where θ lies between 0 and π. So the total mechanical moment in this case is got by integration of $mrun \cos\theta \sin\theta d\theta/2$ between those limits. Thus

$$M = \frac{n}{4} r^2 eH \int_0^{\pi} \cos^2\theta \sin\theta d\theta$$

$$= \frac{1}{6} ner^2 H$$

But (§ 18) the induced diamagnetic moment is

$$G = -\frac{ne^2r^2H}{12m}$$

the meaning of n being altered. Hence in this case also

$$\frac{M}{G} = -\frac{2m}{e}$$

and it appears that the imposition of the induced speed upon the speed v existing in paramagnetism does not alter the ratio. Thus the exact value of the ratio, in paramagnetism and diamagnetism alike, is numerically $2m/e$.

This is in so far as the *orbital circulation* is concerned.

31. The Precessional Moment of Momentum.—If a body is spinning with angular velocity ω, right-handedly round a horizontal north pointing axis, the moment of momentum, or angular momentum, is $I\omega$, where I is the moment of inertia round the axis of spin. If now we impose a torque (angular force) right-handedly round an east pointing axis, we are attempting to produce moment of momentum round that axis, and this is resisted by inertia. Linear force is equal to the product of mass and linear acceleration: and the mass is the measure of the linear inertia which opposes the linear change of momentum. Similarly angular force (torque) is opposed by the angular inertia (moment of inertia). That is to say, angular inertia tends to maintain unaltered the moment of momentum round the north pointing axis. And the easterly torque tending to produce easterly moment of momentum is strongly opposed by the northerly angular inertia. The effect which the torque is tending to produce can be produced indirectly, much more readily, if the axis of spin turns eastwards in space so as to give an eastward component to the originally northerly moment of momentum. This implies that rotation round a downward axis has entered in. And, since the axis of torque is perpendicular to the axis of spin, the rate of spin in the body remains constant, and so the angular momentum due to spin, $I\omega$, remains constant in magnitude while it alters in direction. Further, it turns or precesses at a constant angular rate, if the torque remains constant. That is to say, the easterly change

of angular momentum, which is equal to the northerly angular momentum multiplied by its angular rate of precession towards the east, remains constant. But that easterly rate of change of angular momentum is just the torque T. So

$$T = I\omega\Omega$$

In the case of an orbital electron, the moment of inertia round the (right-handedly directed) normal to the plane of the orbit is ma^2 if the orbit is a circle of radius a. If it be elliptic, a represents the mean radius in the rapid orbital revolution.

From the fact that the magnetic moment of any small circuit of area a, carrying a current i, is ia, it follows that the magnetic moment due to a circular circuit of radius a is the same as that

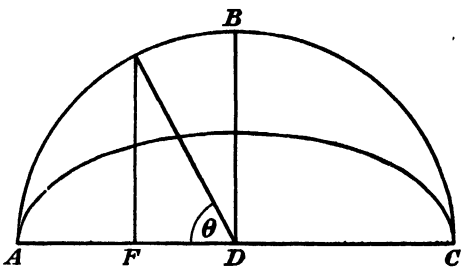

FIG. 10.

due to its three rectangular projections, regarded as independent circuits each carrying the same current i.

So the component of magnetic moment in any direction inclined at an angle ϕ to the normal to the circular orbit is $ia\cos\phi$, and is the same as that due to an elliptic orbit, the projection of the circle, placed normally to the given direction. The condition that the current is the same requires that this orbit, if it is the path of an electron, shall be described in the same period as that required for the circular orbit. In other words, the average angular velocity, or the periodic time, must be the same.

But, if the mechanical conditions are to be equivalent in an elliptic path and a circular path, the moment of inertia of the circulating electron must also be the same in both. To find the condition for this, let F (Fig. 10) be a focus of an ellipse, whose ordinates are less, in a constant ratio, b/a, than those of the circle ABC, at the same abscissæ. The semi-axis major being

equal to the radius of the circle, the periodic time will be the same, as is well known, if the electron describes the ellipse under a central attraction to F which varies inversely as the square of the distance of the electron from F. We have now to determine whether the moments of inertia are equal or not.

The ellipse may be regarded as the projection of the circle turned round AC through an angle ϕ. If a be the semi-axis major, b is the semi-axis minor, and we have $b = a\cos\phi$. Also $FD = ae$, where e is the eccentricity; and so $e = \cos\theta$.

Now the moment of inertia of the electron in the circular orbit, round an axis, through D, perpendicular to the plane of the paper, being I, the moment of inertia, round the same axis, of the electron describing the ellipse, is $I\cos^2\phi$. Hence its moment of inertia round a parallel axis through F is

$$I\,.\cos^2\phi + m\,.\,a^2e^2 = ma^2(\cos^2\phi + \cos^2\theta)$$

But $b^2/a^2 = \cos^2\phi$ is equal to $1 - e^2$. Therefore $\cos^2\phi = \sin^2\theta$, and so the moment of inertia in a circular electronic orbit is equal to the moment of inertia in any elliptic orbit in the same plane, and having its mean radius (or semi-axis major) equal to the radius of the circle, the moment being taken with reference to the centre of force in all cases.

Strictly speaking, this centre of force is the centre of inertia of the electron and the remainder of the atom or molecule. But the difference is negligible, as also are the magnetic effects due to the motion of the positive charge of the remainder in general.

32. The Precessional Moment of Momentum (continued).—Since, therefore, for our present purposes, we can replace any actual elliptic orbit by the corresponding circular one, we can now continue the treatment of circular orbits alone without loss of generality. Undoubtedly, in the elliptic orbit, the moments of inertia are not equal about the major and minor axes, but these conditions are completely averaged by the random spatial distribution of the axes throughout the immense number of molecules per unit volume.

Let the line ns, Fig. 11, which is the axis of revolution in an equivalent circular orbit of radius r, shown in perspective in the diagram, be inclined at an angle θ to the direction of the field H. The moment of momentum round ns is mvr. Therefore, round

the direction of the field, it is $mvr \cos\theta$. The effect of this has been already discussed (§ 30). But the torque is round $AC(\perp AB)$ and so the precession, Ω, is round AB. The moment of inertia round the axis being mr^2, the moment round AB is $mr^2/2$; therefore the moment of momentum round AB is

$$\frac{1}{2}mr^2 . \Omega = \frac{Her^2}{4} \sin\theta$$

and the component of this round the field is

$$Her^2 \sin^2\theta/4$$

The other component vanishes in the summation, since positive

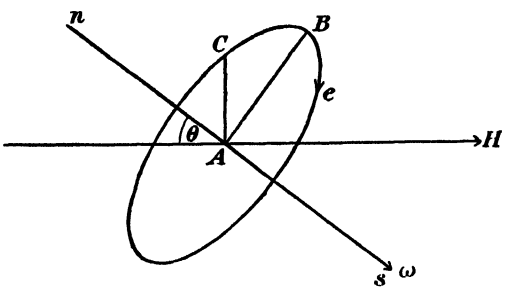

FIG. 11.

and negative values of θ occur equally. And the total number of orbits being n per unit volume, the total precessional moment of momentum is equal to the integral from 0 to π of $nHer^2 \sin^3\theta d\theta/8$. Therefore its value is

$$M = nHer^2/6$$

and is negligible in comparison with the paramagnetic circulatory moment of momentum of magnitude, $-nvrm$, discussed in § 30. It is to be noted that the torque is reversed when the spin is reversed, so that the precession is not reversed.

And, since the rate of precession is quite independent of the speed in the orbit, *this expression also represents the moment of momentum which arises precessionally in the case of diamagnetism.* In this case the integration extends only from 0 to $\pi/2$; but, all axes lying within a hemisphere, n replaces the former $n/2$, so that the result is unaltered.

33. The Ratios of the Moments.—The paramagnetic case has been considered in § 30. And the precessional diamagnetic moment has been dealt with in § 21. The latter has the same value

$$G = -\frac{He^2r^2n}{12m}$$

as the diamagnetic moment arising from the inductive effect (§ 18), and we have just found that the mechanical momentum has, in this case, the value

$$M = \frac{neHr^2}{6m}$$

Consequently the ratio of the mechanical momentum to the magnetic momentum,

$$\frac{M}{G} = -\frac{2m}{e}$$

has the same constant value in the cases of all the component contributions—paramagnetic, induced diamagnetic, and gyrostatic diamagnetic.

In the case of paramagnetism, it is the mechanical moment which is negative; in the other cases, both of which occur in diamagnetism, it is the magnetic moment which is negative. All really occur in paramagnetism, but the inductive and gyroscopic effects are negligible. It is important to note, however, that precessions must cease if complete saturation is attained, for the axes are then parallel to the field, so that the torque ceases. At the same time the inductive M becomes $nmru$, and the inductive G becomes $-neru/2$, the ratio being still unaltered. The expressions for the inductive and gyrostatic quantities, M and G, have all been calculated on the assumption of random distribution before the field was applied and after. The justification of this lies in the fact that the gyrostatic action preserves the initial orientation of the orbits to the field.

34. The Gyromagnetic Anomaly.—The mechanical and magnetic moments have, in the preceding discussions, been regarded as resident in the molecules. But we have just seen that changes in them must take place if magnetization is effected. If

the magnetization is accompanied by transference of angular momentum from the orbits to the molecules by collisions, and if the transferred momentum is measured experimentally by its effect on the magnetized body, while the magnetic moment is also measured experimentally, it does not necessarily follow that the calculated ratio will be found. We must rather look to the experiment to furnish information regarding the nature of the actions. The observed ratio does not agree with the theoretical ratio. This is known as the *gyromagnetic anomaly*.

35. The Experimental Results.—The prediction may be tested in two ways, both of which have been employed. In one, the magnetized body may be suspended so as to be free to rotate around an axis lying in the direction of the field which is to be applied. The magnetic moment produced by the field can be measured by one of the usual methods; and the angular momentum, if any, which is communicated to the body, can be measured by the ensuing twist of the suspending fibre.

When the field is imposed, the torque, as we have seen, tends to turn over the axis of the paramagnetic molecules into coincidence with the direction of the field. But the effect of the orbital spin of the electrons gives rise instead, to precession round that direction, and no paramagnetic moment is produced. It can only be produced if the precession is opposed. If precession be entirely prevented, the molecule is quite mobile, and the work done by the field appears in the form of kinetic energy of rotation round an axis perpendicular to the field. Steady opposition to the precession gives rise to continuous orientation of the molecule until, when the precession vanishes, the magnetic axis is parallel to the field, and the change of moment is a maximum. At the same time the moment of momentum of the molecule round the direction of the field vanishes. But conservation of moment of momentum requires that it shall have passed to the system which has opposed the precession. And, if the opposition has been produced by molecular impacts, the angular momentum must now reside in the molecular system. If that system be solid, the body must acquire the momentum, and will twist the suspending fibre to an extent from which the magnitude of the momentum, so communicated, may be found.

Great difficulty arises in the experimental determination from

the extreme smallness of the effect to be observed. Very slight sources of error may entirely mask the result. The most reliable results found in the case of strongly paramagnetic substances indicate an angular momentum which is only one half of the theoretical amount.

In the other method the body is rotated about an axis. $I\omega$ is the corresponding momentum, where I is the moment of inertia, and ω is the angular velocity. The induced magnetic moment is measured experimentally; and here also the experimental difficulties are excessive. The theory indicates that, with 10 revolutions per second, the field produced would be little more than $7(10)^{-5}$ gauss. Nevertheless the observations show that, within experimental errors, the magnetic moment produced is twice as great as that which is indicated by theory. The two methods lead to the same result.

In the case of diamagnetism, the experimental conditions are much worse in consequence of the feebleness of the effects. The results are said to indicate the absence of any diamagnetic moment arising from rotation. If this be correct it follows that no bodily rotation should result from the application of a magnetic field.

36. The Nature of the Actions.—In the absence of a magnetic field, a diamagnetic body at rest has no magnetic moment and no mechanical moment. If a rotation is superposed, the effect will depend altogether on the nature of the connection existing between the molecules which constitute the framework of the body and the sub-molecular, or sub-atomic, mechanism which gives rise to the magnetic effects. If the latter be absolutely free, rotation will neither give rise to precession of electronic orbits nor to resultant circulation of the electrons in the orbits, and so no magnetic moment will arise. If, on the contrary, the rotational motion be communicated to the mechanism of balanced electronic orbits, the balance will be destroyed, and a diamagnetic moment will result.

In diamagnetism the two constituents, circulatory and precessional, of the diamagnetic moment are equal, as also are the two corresponding constituents of the mechanical momentum. Both constituents are originated by the superposed field. And, if the gyratory motion alone were destroyed by molecular impacts, the corresponding angular momentum being transferred to the

neutral molecules, the ratio of the mechanical and the diamagnetic moments retains the value $-2m/e$. The experimental results seem to indicate that in paramagnetism half of the mechanical momentum is transferred to the framework; while in diamagnetism none is transferred to it; the untransferred parts must pass to some different system—of radiational or other type. In the absence of more complete experimental evidence bearing on the nature of the action, the whole matter is purely speculative.

The theory which has just been discussed postulates the existence of a continuous range of moments and momenta. In the Quantum theory (Chap. IX), on the other hand, discrete values alone enter in, and these are integral multiples of fundamental units. Richardson has shown that it may be possible to account for the paramagnetic anomaly on that theory if the nucleus and the electron alike move in quantum orbits—provided that their momenta are *oppositely* directed. If M_e and M_n, G_e and G_n respectively represent the electronic and nuclear momenta and moments, while M and G are the resultant quantities, we have

$$M = M_n + M_e$$
$$G = a(\xi M_n + M_e)$$

where $a = e/2m$, and ξa is the corresponding ratio for the nuclear charge and mass, so that ξ is of the order 10^{-3}. Hence we get approximately

$$\frac{M}{G} = \frac{2m}{e} \cdot \frac{M_n + M_e}{M_e}$$

which gives the observed paramagnetic ratio if $M_e = 2$, $M_n = -1$. But the postulate of a negative nuclear momentum is very extreme.

The quantum theory cannot at present give a real explanation of the phenomenon, for it is founded on unexplained arbitrary hypotheses, although these give results which are in wonderful agreement with observation, and must be regarded as having to a considerable extent some real, though at present unknown basis. It is possible that the basis may be revealed through the methods of wave mechanics.

The experimental verification of the prediction that rotation of an unmagnetized body, whose molecules contain electronic orbits, will impose upon it a reverse magnetic moment, is of great

interest in connection with the earth's magnetization which exhibits the existence of a south magnetic pole in the north polar region. The direction of the magnetization is correctly indicated, but the actual magnitude largely exceeds the amount which corresponds to known atomic structure.

37. Dimensional Effects. Magnetostriction.—A molar mechanical effect of magnetization has been made evident in the course of the preceding section; but, long before that one was observed, Joule had shown that an iron rod increased in length when magnetized longitudinally, and that it contracted laterally

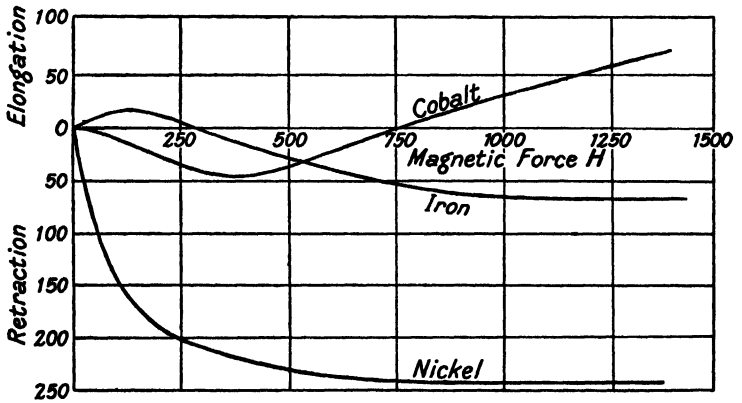

FIG. 12.

to an extent which seemed to indicate no change of volume. These statements were subsequently found to require modification in dependence on the intensity of the applied field. This increase of length attains a maximum, diminishes to zero, becomes negative, and seems to tend towards a maximum contraction at very high fields.

In the case of cobalt, shrinkage occurs at low fields, reaches a maximum, diminishes, reverses, and the elongation then increases as the field is increased. The sequence is of opposite type to that exhibited by iron. In the case of nickel, contraction is always evident and increases towards a maximum as the field increases in strength. Fig. 12 exhibits Bidwell's results. Knott showed that Joule's result regarding no change of volume in iron was due to the special magnitude of the field with which he worked, and he

established the fact that change of volume actually occurs, with a reversal of sign at an intermediate field.

From these data it may be predicted that if, for example, an iron rod be magnetized spirally, an increase of length of the material will occur in the direction of the spiral in weak fields, along with a shrinkage in directions perpendicular to the spiral. But these effects are precisely those which accompany twist of the rod. Hence a twist of the rod should be originated by the magnetization. This effect, having been first investigated by G. Weidemann, is known as the *Weidemann Effect*. The spiral magnetization can be brought about by causing a current of electricity to flow along the rod when it is longitudinally magnetized. This superposes circular magnetization upon the longitudinal magnetization, and the resultant is spiral magnetization. Alternatively, the current may be made to flow along a tube surrounding the rod.

Phenomena of these types are known as *Magnetostrictive Effects*. From them we may predict by means of the conditions for stability the existence of converse effects. But there we have to consider very carefully the nature of the corresponding relationships. If, in the case of two physically related quantities A and B, an increase of A gives rise to an increase of B, an increase of B must give rise to a decrease of A, *provided that A and B are directly related*. But, A and C being directly related, as also C and B, if an increase of A gives rise to an increase of C which causes an increase of B, that increase of B will produce a decrease of C, and that decrease of C will give rise to an increase of A. If A and B alone be observed, it will appear as if increase of A produces an increase of B, and that an increase of B also produces an increase of A. It does not follow that A and B are unstably related, even within limits as to the magnitude of the resultant changes. But if instability does not intervene, the conclusion is that there is at least one intermediate connection and, in any case, an odd number of intermediate actions.

Now in iron, in weak fields, increase of magnetization gives rise to increase of length; and an independent increase of length in a magnetic field causes an increase of magnetization. We must therefore conclude either that instability has been effective or that the dimensional change is not a direct effect of the magnetization.

The solution of this problem can perhaps only be attempted through experimental tests on the most highly homogeneous crystals within the range in which magnetization is proportional to the strength of the field. But the dimensional changes in these feeble fields must be almost vanishingly small. It may be questioned whether experiments with a rod (congeries of crystals) supply the same test. Nevertheless they would be important.

38. Maxwell's Stresses.—Consider the ether contained between two infinite parallel planes, at unit distance apart, and subject to a uniform magnetic field of intensity H directed along their normal. We shall presume that this portion of the ether acquires zero permeability with no other change of properties. It is then diamagnetic towards the external ether, and must exhibit diamagnetic polarization of intensity $I = H/4\pi$, since lines of force are 4π times more numerous than lines of induction. Therefore the energy contained in unit volume, being equal to the work done in separating the north and south magnetisms, is

$$\int HdI = \frac{H^2}{8\pi}.$$

This is dependent on H alone, and is equal to the force exerted on the planes, per unit area, since force is space rate of variation of energy. Work is done in removing the energy from the slab, and therefore the force is an inward pressure acting so as to refill the slab with the energy. It acts similarly across any surface of separation between ordinary and non-permeable ether. In particular it would act inwards over the surface of a small sphere of zero permeability. The expression $H^2/8\pi$ therefore indicates a hydrostatic pressure existing in normal ether wherever it is traversed by a field of force.

Now let the ether within the slab acquire positive susceptibility. The extra work, in excess of that needed to establish the constant field, is that of turning the molecular magnets into parallelism with the field, that is,

$$H\int nmdl = H\int dI = H\int \frac{dB}{4\pi} = \frac{HB}{4\pi}$$

where n is the number of magnets per unit volume, m is the pole

strength of each, and l is the length of each. Thus there exists a tension, along the lines of induction, tending to destroy the magnetization, that is, to make the lines shrink; and its magnitude is $HB/4\pi$. To obtain the total stress in the medium we have to add to this the hydrostatic pressure $H^2/8\pi$, as found above.

To find the more general expressions for a crystalline medium when B and H are not co-directed, but are inclined at an angle 2ϕ to each other, we can break up these two vectors into their components parallel respectively to the internal and external bisectors of the angle between them. Let these be B_i and B_e, H_i, and H_e. We thus get the four products B_iH_i, B_eH_e, B_iH_e, B_eH_i. The component vectors B_i and H_i are co-directed, while B_e and H_e are oppositely directed. Consequently the product B_iH_i has the same, while the product B_eH_e has the opposite, interpretation to that found for the BH product when B and H coincided in direction. Therefore we have now (1) a tension $B_iH_i/4\pi = BH\cos^2\phi/4\pi$ along the internal bisector, and (2) a pressure $B_eH_e/4\pi = BH\sin^2\phi/4\pi$ along the external bisector.

The vector products B_iH_e and B_eH_i are similar to each other in every way, and have the same numerical value. Now the $BH\sin\phi\cos\phi$ product of two mutually perpendicular vectors must be similarly related to both. It must therefore be a vector mutually perpendicular to both. And, in the present case, it must represent the axis of the mechanical torque which is acting so as to turn the molecular magnets, per unit volume, into alignment with the field, and which is opposed by the crystalline constraints. Thus we have (3) a torque of magnitude

$$(B_iH_e + B_eH_i)/4\pi = BH\sin 2\phi/4\pi,$$

tending to turn the crystal round so that a principal axis shall coincide with the direction of H.

If the medium be isotropic, B and H are co-directed so that $\phi = 0$. In that case a rod longitudinally magnetized tends to contract because of the longitudinal tension; and the transverse pressure, acting outwards if there is no field outside it, or a weaker field, also tends to induce longitudinal contraction. But if the medium is randomly crystalline, that is to say randomly isotropic, which is the case with iron, nickel, and cobalt, we cannot postulate so simple a condition; for the solid framework of the metal is a congeries of variously orientated crystals, to each of

which individually the more complicated expressions involving ϕ have to be applied.

39. Theoretical Treatment (continued).—In the case just mentioned the longitudinal tension is lessened in magnitude, and is moreover inclined at the angle ϕ to the direction of the field presumed to be parallel to the length of the rod. But the lessening of lengthwise contraction, due to this cause, is in part compensated by the transverse pressure involving $\sin^2 \phi$. We are therefore compelled to look to the effect of the torque, which involves $\sin 2\phi$ for the longitudinal expansions which are apparent in the cases of iron and cobalt (Fig. 12). And the conditions are complicated since the initial effects are opposite in the two metals.

Both metals (and also nickel, which acts so differently) crystallize on the cubic system, the formation being either face-centred or body-centred. Now in the homogeneous face-centred or body-centred cubic arrangements, the axes of maximum permeability are the body diagonals. These are therefore the crystalline directions which the torque tends to turn into parallelism to the field. Thus the tendency is towards an increase of linear spacing of the constituent crystals in the direction of the field, with some transverse contraction, if the cubic form is geometrically complete.

But if the cubic crystals have symmetrically truncated corners, in particular if the truncation is complete enough to make the external form octohedral, the shortest crystalline dimensions tend to longitudinal orientation. In such a case contraction must occur at all field strengths and tend rapidly to a maximum—as in the case of nickel.

The existence of crystalline flaws might also have a great effect. In spite of the fact that the body diagonals are the directions of stable magnetization, and so of maximum permeability, the existence of incipient flaws in planes perpendicular to these directions might make the direction of induction more nearly parallel on the average to the face diagonals. In this way longitudinal dilatation could be lessened.

Again, the values of the elastic constants, according as there is random orientation of the constituent crystals, or more nearly co-orientation of them under the action of the torque, require consideration. Thus if, under co-orientation of body diagonals of

complete cubes with the field, the linear rigidity in that direction were great enough in comparison with its value in a transverse direction, the final effect might be in the sense of elongation.

Further, although X-ray analysis indicates that in each of these three metals the crystalline structure is that of the face-centred or body-centred cubic arrangement, it takes no cognizance of magnetic quality. It is not inconceivable that the magnetic moment of the molecules at face centres is different from that of those

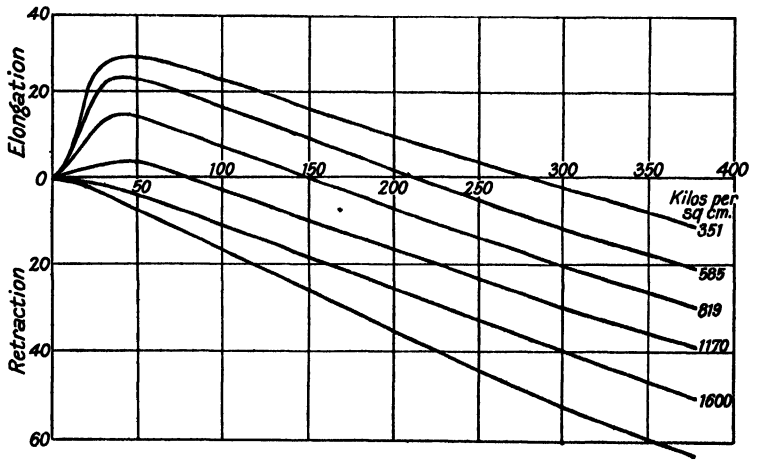

FIG. 13.

at cube corners. In other words, a face-centred cubic molecular arrangement may possibly not be of that type magnetically. There are many unknown quantities whose determination is necessary before the theory of magnetostriction can be really tested. But the theoretical possibilities are perhaps wide enough for its applicability. The fact that independent extension of an iron wire, by an external load, wipes out the initial elongation which, under no load, is evident in weak fields, is in direct relation to the theoretical results (Fig. 13). For the widening of the longitudinal spacing by the load gives less occasion for work of extension by the torque.

CHAPTER IV

RANDOM MAGNETIC AGGREGATES

40. Paramagnetic Gases and Solutions.—Conditions of magnetization take their simplest form in the cases of a gas, or vapour, and a solution. The molecules are in general outside the range of appreciable intermolecular action, so that the external field alone has to be considered. In the case of solutions any influence of the magnetic quality of the solvent must be allowed for.

Let n be the number of molecules, each of magnetic moment μ, in unit volume of the substance, while $\nu = dn/d\omega$ is the number of north poles taken per unit solid angle in that volume. Before the external field H is applied, the distribution of the magnetic axes is uniform, so that ν is constant. But, when H acts, the axes tend to alignment with the field, and are only kept therefrom in consequence of the thermal motions existent at the absolute temperature T. The result is that ν increases in the directions adjoining that of H and decreases in more remote directions. It is constant only in directions which make a constant angle θ with H.

Each molecule has rotational freedom about lines perpendicular to its axis, and the effect of the application of H is, as we have seen, to cause the axis to precess round H at constant θ. The application of H causes a slight change in θ and brings in a slight change of the component of the magnetic moment parallel to H. That component being $\mu \cos\theta$, the work done by H is $Hd(\mu \cos\theta)$, and it is this which appears as precessional kinetic energy. Thus per unit solid angle in the near neighbourhood of θ the work is $\nu\mu Hd(\cos\theta)$. But, if ν increases by $d\nu$ at constant temperature, energy enters, per unit solid angle, to the amount $d\nu \,.\, kT$ where kT is the mean energy per degree of freedom at the temperature T.

At constant temperature the average translational energy

remains constant per molecule. But part of it is spent in opposing the precessional action of the field, and must be compensated by the influx due to the increase of ν. Hence

$$kTd\nu = \nu\mu Hd(\cos\theta)$$

from which we find

$$\nu = \nu_0 \epsilon^{\frac{\mu H\cos\theta}{kT}},$$

whence we can obtain the magnetic moment per molecule.

41. The Magnetic Moment as dependent on Temperature and Field. Langevin's Formula.—From the preceding expression for ν, with $d\omega = 2\pi\sin\theta d\theta$, we get, putting $\mu H/kT = P$,

$$n = -2\pi\nu_0\int_{-1}^{+1}\epsilon^{P\cos\theta}d(\cos\theta)$$

$$= -2\pi\frac{\nu_0}{P}(\epsilon^P - \epsilon^{-P}) = -\frac{4\pi\nu_0}{P}\sinh P$$

Now the magnetic moment per unit volume is

$$I = \int\mu\cos\theta\,.\,dn$$

$$= \int_0^\pi \mu\cos\theta\,.\,\nu_0\epsilon^{P\cos\theta}\,.\,2\pi\sin\theta d\theta$$

$$= -\frac{2\pi\mu\nu_0}{P^2}\int_{-P}^{+P} x\,.\,e^x\,.\,dx$$

$$= -\frac{2\pi\mu\nu_0}{P^2}\left\{\left[x\epsilon^x\right]_{-P}^{+P} - \left[e^x\right]_{-P}^{+P}\right\}$$

$$= -\frac{2\pi\mu\nu_0}{P^2}\left\{P(e^P + \epsilon^{-P}) - (\epsilon^P - \epsilon^{-P})\right\}$$

$$= -\frac{4\pi\mu\nu_0}{P^2}\left\{P\cosh P - \sinh P\right\}$$

$$= \mu n\left(\coth P - \frac{1}{P}\right)$$

an expression first given by Langevin. Dividing it by n we obtain the mean molecular moment. If

$$P = \frac{\mu H}{kT}$$

is large, the bracketed quantity becomes $1 - 1/P$; if P is small it becomes $P/3 - P^2/45$.

Thus we have

$$I = \mu n\left(1 - \frac{1}{P}\right)$$

$$I = \mu n\frac{P}{3}\left(1 - \frac{P}{15}\right)$$

according as μH is large or small relatively to kT. The curve in Fig. 14 (p. 59) represents the values of $\coth P - 1/P$ for values of P given as abscissæ. The ordinates asymptotically approach the value unity as P increases without limit. This condition can only be approximated to at very low temperatures.

In ordinary paramagnetics I is found to be proportional to H, so that the first term is sufficient, and we have

$$I = \mu n\frac{P}{3} = \mu n \cdot \frac{\mu H}{3kT}$$

Even in the case of soft iron at ordinary temperature, P would be only about 0·05. The second term would not become appreciable unless T approached 50° abs. But the theory is inapplicable to iron because of the existence of the internal field. Nevertheless, as the internal and external fields are found to be of the same order of magnitude, the calculation indicates with sufficient correctness the restriction which is imposed on the available range of the curve at usual temperatures.

42. Paramagnetic Susceptibility: Curie's Law, Diamagnetics.—The susceptibility I/H is given by

$$\chi_v = \frac{\mu^2 n}{3kT}$$

and therefore varies in inverse proportion to the absolute tem-

perature. This is the volume susceptibility. The mass, or specific, susceptibility is

$$\chi = \frac{\mu^2 n}{3kpT}$$

where ρ is the density; and this quantity, as Curie pointed out, is of greater physical importance than χ_v. Also

$$\chi_m = \chi \cdot m,$$

where m is the molecular mass, is the true molecular susceptibility at temperature T.

The important theoretical result here is that the susceptibility, when internal fields are negligible, should vary inversely as the absolute temperature.

Curie specially tested this point, using data of other observers as well as his own observations, and he proved that the general law

$$\chi = \frac{C}{T}$$

where C is a constant, holds well for oxygen and air, and for various paramagnetic solutions, as well as approximately for some solid salts. This law is known as Curie's Law.

Diamagnetic susceptibility, according to the general theory of diamagnetism already discussed, should be independent of the field (single crystals excepted in general), and should be independent of the temperature. Curie found this to be usually the case. Bismuth exhibits a peculiarity.

Ferromagnetic substances become paramagnetic at sufficiently high temperature.

43. The Internal Field: Weiss's Treatment.—Weiss introduced into the theory of paramagnetism the consideration of the effect of the internal field when appreciable. He postulated that the internal field was proportional to the intensity I of magnetization, so that the total field is

$$H = H_0 + NI$$

where H_0 is the external field and N is a constant.

So far as the action on any one molecule is concerned, it is evident that the internal field is in a continuous state of flicker, both as to magnitude and direction, in consequence of the variable location and orientation of the surrounding molecules in its immediate neighbourhood. Thus N represents the average of a variable magnitude, which remains constant in virtue of the enormous number of molecules in even a small volume.

In the expressions (§ 41)

$$I = \mu n\left(\coth P - \frac{1}{P}\right)$$

$$P = \frac{\mu H}{kT}$$

we must now write the above value of H.

In particular, if no external field is acting, we have

$$P = \frac{\mu N}{kT} \cdot I$$

Writing $\mu n = I_0$, the saturation intensity, we get two values of I

$$I = I_0\left(\coth P - \frac{1}{P}\right)$$

$$I = I_0 \frac{kTnP}{I_0^2 N}$$

which are shown graphically in Fig. 14, I/I_0 being the ordinate at the abscissa P. The point of intersection, B, of the two graphs gives the value of P at which the internal field would, by itself, give a condition of stable magnetization.

Now P is small, except at very low temperature, even in very large fields. And k is an energy value per molecule. The value per unit volume is kn and so the value per gramme-molecule is $kn\omega/\rho$ where ω is the atomic weight, and ρ is the density. But this is the value of R, the gas constant, so that

$$P = \frac{\mu H}{kT} = \frac{\mu n \omega H}{\rho RT} = I_0 \frac{\omega}{\rho RT} H$$

In the case of iron, $I_0 = 1{,}700$, $\omega = 55{\cdot}8$, $\rho = 7{\cdot}8$, and the

value of R is $8 \cdot 3(10)^7$. So, even if we take $H = 10^5$, with $T = 300$ the value of P is only about $1/20$. So the second term in the approximation for I at small values of P can be neglected. Under usual conditions, therefore, we get

$$I = I_0 \frac{kTn}{I_0^2 N} \cdot P = \frac{I_0}{3} \cdot P$$

as the value of the spontaneous magnetization with zero external field.

This indicates a limiting temperature

$$T = \frac{I_0^2 N}{3kn} = \frac{I_0^2 N\omega}{3R\rho}$$

at which spontaneous magnetization of a paramagnetic substance just becomes possible. At higher temperatures the effect of

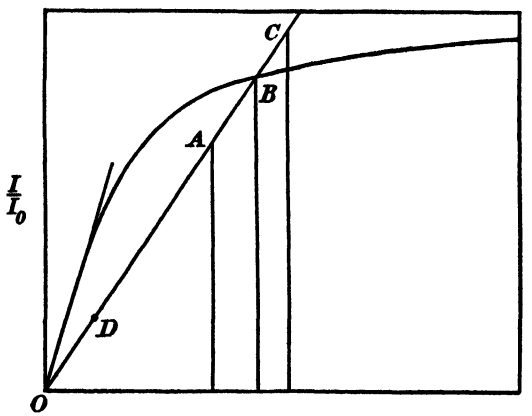

FIG. 14.

collisions is sufficient to overpower the tendency towards an average alignment, spontaneously, of the magnetic molecules in one preponderant direction. At lower temperatures the slope of the straight line OC in Fig. 14 is such that it cuts Langevin's curve, in general, at a point for which the approximations cannot be used.

But it is easy to see that the state indicated by B is stable under the spontaneous magnetization. For if, in some region of the substance, a group of molecules assumes the state A, the

local magnetization is less than that which corresponds to equilibrium. So the magnetization, and consequently the field, and the value of P, tend to increase until (T remaining constant) the point B in the diagram is reached. Similarly, if a local C-state originates, the local magnetization is greater than its equilibrium value, as indicated by Langevin's curve. So the magnetization and the field tend to decrease until B is reached.

A purely random molecular condition is also unstable. This state is indicated by the origin O, at which Langevin's equilibrium curve and the straight line OC also intersect. For, if a local state D is set up, the magnetization is less than the equilibrium value; and so it, and consequently the internal field, tend to increase until B is reached.

44. Weiss's Modified Law of Curie. The Curie Point. —The approximate relation

$$I = I_0 \frac{P}{3} = I_0^2 \frac{H + NI}{3R\rho T}\omega$$

gives

$$I = I_0 \frac{\dfrac{\mu H}{3kT}}{1 - \dfrac{\mu I_0 N}{3kT}}$$

$$= \frac{I_0^2 H \cdot \dfrac{\omega}{3R\rho}}{T - I_0^2 N \dfrac{\omega}{3R\rho}}$$

$$= \frac{I_0^2 \dfrac{\omega}{\rho} H}{3R(T - \Theta)}$$

where

$$\Theta = \frac{I_0^2 \omega N}{3R\rho}$$

is the limiting temperature, the meaning of which has just been

considered. It appears therefore that the volume susceptibility I/H is given by the simple expression,

$$\chi_v = \frac{C}{T - \Theta}$$

where $C = I_0^2\omega/3R\rho$ is the same constant $(\mu^2\eta/3k\rho)$ as that used in the expression for Curie's Law. It is known as Curie's Constant. From χ_v we can get χ_ω the susceptibility per gramme-molecule, through multiplication by the volume, ω/ρ, of the gramme-molecule. Thus

$$\chi_\omega = \frac{C_\omega}{T - \Theta}$$

where

$$C_\omega = {}_\omega I_0^2/3R, \quad \Theta = {}_\omega I_0^2 N\rho/3R\omega$$

The law for χ constitutes a generalization, made by Weiss, of Curie's Law. It states that *the susceptibility is inversely proportional to the temperature measured from the Curie Point.* This is known as Weiss's Law.

Many substances are found to obey the law closely, including substances which, below the Curie Point are ferromagnetic. Sometimes Θ is found to be negative, which implies that the internal field is negative. Observations of the values of C and Θ make determinations of I_0 and N possible. Weiss's Law cannot be expected to hold exactly as T approaches Θ, for the approximate expression for $(\coth P - 1/P)$, which was used in its deduction, ceases then to be applicable.

The magnetic action in random crystalline aggregates requires, for its discussion, consideration of the action in a homogeneous crystalline medium. Therefore its consideration must be deferred meanwhile, except in so far as each constituent crystal may be imagined to be replaced by a single equivalent magnet. In that case the conditions postulated in the deduction of Weiss's Law can be regarded as at least approximately applicable.

The value of the "spontaneous saturation," that is of the magnetization under the intrinsic field alone, depends on the temperature. It is always appreciably less than the true saturation value, unless the temperature be very low: but a very high

intrinsic field, should such exist, would produce a similar result. In subtances for which Θ is negative, it is evidently not possible that magnetization may occur apart from the action of an external field. The presence of the internal field is then only revealed by its influence upon the effect of an externally applied field.

CHAPTER V

THE WORK OF EWING, WEISS, AND OTHERS

45. The Elimination of Subsidiary Postulates.—The introduction of subsidiary postulates in the development of a theory is often necessary. This sometimes occurs because of mathematical difficulties, for the theoretical treatment may, in its most general form, be incapable of further advance by available mathematical methods. A simplifying postulate has then to be added, and the results of the theory as developed under its domination must be submitted specially to experimental test so as to determine whether it is justified or not. In other cases, the general theory may give different types of results, all of which may be possible in a corresponding variety of circumstances. But the physical phenomena may indicate that one only of these types is found to exist. The subsequent mathematical development must then be followed, in the limited direction, so as to explain established phenomena or predict new ones. But the physicist will also seek to inquire into the physical origin which lies behind the limitation.

We have seen, in the preceding chapter, that Weber never deviated from the generality and simplicity of the postulate of Ampèrean molecular electric circuits, in which permanent or transient currents could flow. When the currents were permanent, the molecule was endowed with permanent magnetic quality. It became effectively a molecular magnet of constant magnetic moment; and a random congeries of these molecules would form a paramagnetic body. Nor did he deviate from the generality and simplicity of his second, and only other, postulate of directional magnetic control of each molecule by the magnetic influence upon it of the surrounding molecules. No other action was to be considered. He reiterated this assertion, and applied it also in the case of diamagnetism, which he held to be due to transient

molecular currents induced, in accordance with Faraday's laws, by superposed magnetic fields.

He worked out the law of mutual force between two adjacent molecules, regarded as small magnets, and gave thereby a beautiful explanation of the characteristically different actions of a diamagnetic rod and a paramagnetic rod placed in an external magnetic field.

Further, he applied these ideas to explain the course of magnetization of a magnetic mass placed in a magnetic field increasing from zero. But in this case he did not evaluate the additional magnetic field acting upon any one molecule in consequence of the presence of its magnetic neighbours. He avoided the mathematical difficulties by postulating a simple form for the law of resultant action. And he justified it sufficiently by a comparison of the consequent theoretical law of magnetization with the observed law of magnetization of soft iron. His postulate of a force representable by $D \sin \phi$ (§ 26) was introduced because it expresses the facts (1) that, in any small deviation from stable magnetization, a restoring internal field must act; (2) that, on the average, in a random congeries of magnets, the resultant internal field must be directed oppositely to the external field, whatever be the individual initial direction of stability of one magnet in consequence of the particular positioning and orientations of its neighbours; and (3) that it is as simple a law as any that we can postulate. It is actually simpler than the law which holds in one of the simplest cases of positioning of neighbours which we shall consider in the following chapter. But it served his purpose well in the case of soft iron, to which he limited his investigation. He knew that it could not account for residual magnetization. So did Maxwell, who modified the postulated law so as to compel it to introduce the residual effect; and he modified it in about as simple a way as was possible for his purpose—the correct procedure in hypothetical work. But neither did he, nor Weber, depart from Weber's fundamental view.

That was done by subsequent workers, who, possibly attempting to read a special physical meaning into the postulated law of force, referred to forces, such as friction, of a different type from that of Weber's mutually interacting magnetic fields.

46. Ewing's Work.—The fundamental merit of Ewing's

work lies in his restoration of the de-railed theory to Weber's lines. Good progress was again possible; and a further merit of his work is that he pushed the investigation of the theory and its results well forward. He worked out the law in detail for certain small groups of magnets. Weber had already done that for two magnets, subject to the restriction that their lengths were small relatively to the distance between their centres. This simplified the mathematical expressions. For simplification of this kind Ewing mainly took the other extreme—small pole clearance between nearest neighbours. He exemplified, in these simple cases, the existence of stable and unstable configurations of the magnets. The *necessity* of their existence in purely magnetic interactions was, of course, unavoidably obvious to both Weber and Maxwell. Weber founded on it his discussion of the stable positions of bismuth and iron rods in a magnetic field; and Maxwell expressly introduced it into his postulated expression for the resultant force due to the interaction.

In particular, by consideration of the stable directions of magnetization (parallel to cube edges) of magnets centred at corners of a cube, Ewing showed that the theory was capable of accounting for the large amount of residual magnetization, observed in some specimens of iron. A solid metal is regarded, as Kelvin indicated, as being a random congeries of small crystals. In the case of iron the crystals belong to the body-centred or the face-centred cubic systems, as the more recent X-ray methods of investigation show. We shall find later, nevertheless, that Ewing's model really illustrates the case of iron. The value obtained was considerably greater than any observed value. Ewing's object, however, was to show that a large enough theoretical value was possible. It must be remembered also that, in the calculation, random heat motions are neglected. These tend to scatter the axes of the molecules from the positions of stability, so that the calculated value of the residual magnetization must exceed the actual value. Another feature of Ewing's work was his introduction of the method of studying the phenomena by means of models consisting of pivoted magnets centred on a geometrical lattice. In this way he exhibited the fact that, in the case of a simple cubic lattice, the cubic axes give the directions of stability under the action of the internal field. The models reproduced also the general form of the magnetization curve

(§ 27) indicative of initial stability, intermediate instability, and final stability in largely new orientations. This method maintained avoidance of the evaluation of Weber's law of mutual magnetic interaction, by appeal to the law of working of a postulated approximate model, just as Weber himself and Maxwell avoided it by appeal to a hypothetically approximate formula. The model, moreover, is subject, at least slightly, to interference by its frictional or other non-magnetic actions to which Weber explicitly denied existence in the actual structure.

In Ewing's treatment it gave also ready illustration of other phenomena, such as the effect of stress on the magnetic medium, and the effect of temperature.

Ewing also completed experimentally, independently of Warburg, who slightly anticipated him in point of time, the hysteresis loop (§ 78) which was already present in part in Maxwell's diagram (Fig. 8) as a result of theory. Other points will be discussed later in connection with the true Weberian law of force.

47. Body-centred and Face-centred Cubic Crystals. Residual Magnetization.—It is easy to show that Ewing's result regarding residual magnetization is not essentially altered if we consider the case of crystals (e.g. nickel) in which the ternary axes (cube diagonals) are the directions of stability.

With randomly orientated crystals in a definite external magnetic field, the field is randomly orientated relatively to the crystalline axes regarded as fixed. So, with three mutually rectangular lines of a cubic lattice taken as fixed axes of reference, and with a sphere of unit radius centred at the origin, the end points of randomly orientated radii will indicate the relatively random directions of the given field. The direction (1, 1, 1) is the direction of stability in the positive octant; and a circle on the sphere, surrounding (1, 1, 1), which just touches the co-ordinate planes, has a radius equal to $1/\sqrt{3}$. Also the similar circle which just passes outside the positive octant has a radius equal to $\sqrt{2}/\sqrt{3}$. And any circle intermediate to these is only in part within the positive octant of the sphere.

If the field be strong enough to give practical saturation, all the molecular magnets, whose directions relatively to the axes are included in the positive octant will, when the magnetizing

force is withdrawn, revert to the direction of stability (1, 1, 1). Therefore if M be the magnetic moment of each molecule, any molecule orientated at an angle θ to the direction (1, 1, 1) will contribute a moment $M \cos \theta$ to the resultant residual magnetization. And, if n be the number of molecules per unit volume,

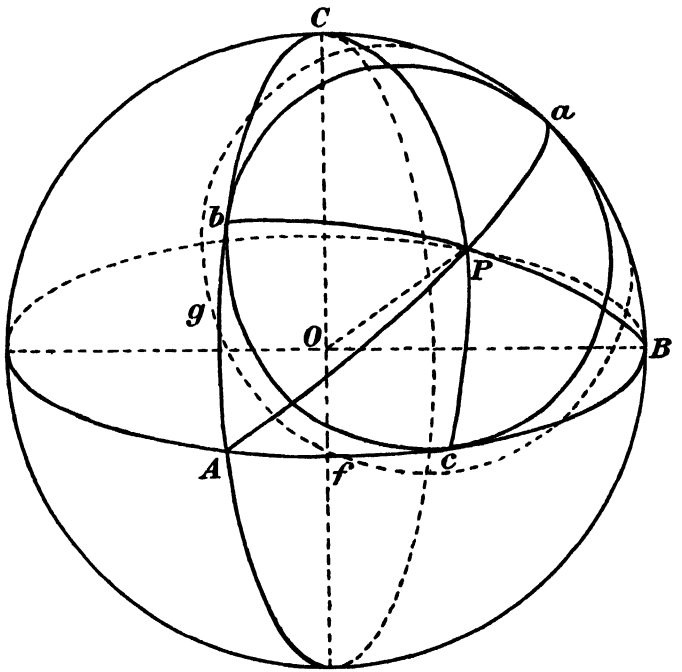

FIG. 15.

the contribution of those magnets, which are inclined within θ and $\theta + d\theta$ to the resultant residual intensity is

$$4Mn \sin \theta \cos \theta d\theta$$

if the circle θ is complete. The factor 4 arises since the whole n end-points are distributed uniformly over the surface of the octant so that the number per unit area is $4\pi n/8$. Now $\sin \theta$ is the radius of the circle; and, as above, the square of the limiting radius for complete circles is $1/3$. So these molecules contribute a residual intensity $2Mn/3$. That circle is shown in Fig. 15, making contact with the great circles, AB, BC, CA, of the octant

(O, A, B, C), at their mid-points, a, b, c. But all magnets whose direction of final stability, after removal of the field, are indicated by points inside the triangular regions, Abc, Bca, Cab, with respect to the point P taken to indicate the direction of this field, revert from the direction OP to their stable directions when the field is removed. If, as before, θ is the angle, which one of these directions makes with OP, the radius, $r = \sin\theta$, of the dotted circle surrounding OP has a value intermediate between $1/\sqrt{3}$ and $\sqrt{2}/\sqrt{3}$. If f and g be the cutting points of that circle with AB and AC respectively, the arcs Af and Ag, each equal to ϕ say, give, as the direction cosines of Of and Og, respectively, the values $(\cos\phi', \sin\phi', 0)$ and $(\cos\phi, 0, \sin\phi)$, ϕ and ϕ' being equal but independent. Therefore the great circle arc fg has the magnitude 2ψ where $\cos\psi = \cos^2\phi$, measured from O. And the arc fg of the small (dotted) circle of radius r, surrounding OP, has an angular measure $\psi/\sin\theta = \psi/r$.

Now consider the surface element $2\pi\sin\theta d\theta$ of the sphere. Only the fraction $6\psi/2\pi r$ of this lies within the octant $OABC$ and so contains points which indicate final stable directions of magnets inclined at the angle θ to the field which had been applied and withdrawn.

Thus we get an addition to the residual moment of amount

$$6Mn\int\psi\cos\theta d\theta$$

the limits of integration being from $\sin\theta = 1/\sqrt{3}$ to $\sin\theta = \sqrt{2}/\sqrt{3}$. We have now to express ψ in terms of θ.

Since

$$\cos\psi = \cos^2\phi = (1 + \cos 2\phi)/2,$$

the plane of the circle being perpendicular to the line $(1, 1, 1)$, we get

$$\sin\phi + \cos\phi = \sqrt{3(1 - r^2)}$$

$$2\sin 2\phi = 2 - 3r^2$$

$$\cos\psi = \tfrac{1}{2}[1 + \sqrt{1 - (2 - 3r^2)^2}]$$

So the addition to the residual magnetic moment is

$$6Mn\int\cos^{-1}\tfrac{1}{2}[1 + \sqrt{1 - (2 - 3r^2)^2}]dr$$

the limits of integration being $r = 1/\sqrt{3}$ and $r = \sqrt{2}/\sqrt{3}$.

The result, obtainable by quadrature, is $(0\cdot1143)Mn$ approximately; and this added to the previously obtained $2Mn/3$ gives

$$\frac{I_r}{I_0} = 0\cdot781$$

as the theoretical value of the ratio of the residual intensity to the saturation intensity.

48. "Temperature" of the Magnetic Elements.—The value just found for the residual ratio of magnetic intensity only applies theoretically in the absence of other fields than the external field; and strictly, only at zero or low absolute temperature. In addition, the demagnetizing action due to crystal boundaries will influence the equilibrium orientations and, consequently, the limits of the integrations. So also will the fields due to neighbouring crystals. There will in general be a field due to near neighbours, although on the whole it will be randomly orientated. Further, temperature being only an average quantity amongst molecules, the influence of abnormally strong collisions may tend to deflect magnets beyond the postulated limiting boundaries when the magnetizing field is removed. Yet the coincidence as regards order of magnitude between the calculated and the observed values indicates that the theoretical treatment gives a good first approximation to the actual conditions.

Now M is, in the treatment, the average value of the effective molecular magnetic moment as subject to temperature or other effects such as those just specified. In the neighbourhood of the transition point, where magnetization is largely destroyed, the scattering of integration limits is preponderantly evident. It is therefore of importance to inquire what influence temperature may have on the value of M *directly*, that is, apart from dimensional changes affecting the average distance of molecules. And, in this connection, it is to be noted that although the influence of temperature on the value of M can be expressed theoretically, no account is taken of the effect of deviations of molecules from their postulated positions on the crystalline lattice, which are only maintained on the average. The influence of the multitudinous crystalline defects from geometrical perfection is also ignored. The normal scientific procedure of postulating ideal

conditions, in order to find out, by the deviations of the actual results from the theoretical results, the real deviations of the actual conditions from those which are assumed, is followed.

From § 43 we see that, in the expression for I, above, in so far as temperature is concerned, I_0 must be multiplied by $P/3$, where P is the value of MH/kT, H being the internal field and M being the saturation moment of a molecular magnet. From the other data given in § 43, with H taken as of the order 10^3, which experiment indicates to be of the order of magnitude of the internal field in an iron crystal (§ 67), the value of P is about 1/2000, so that the multiplier of I_0 is roughly 1/6000.

But, at ordinary temperatures, I_0 is found to differ little from its saturation value, under fields of the order 10^3. Instead of taking $T = 300°$ abs. we would require to take it less than 1° abs. if we were to obtain the observed result.

To change the fractional residual intensity from, say, 0·7 to 0·6, through temperature variations of I_0, would require theoretically a temperature change of 43° C. On the other hand, P being also proportional to H, the sudden drop of the total field from the super-added effects of the external and internal fields to the internal alone, the two being of the same order of magnitude, is capable of accounting for a sufficient *fractional* change in P. But the absolute value of P is not altered in essential magnitude.

Therefore we have an indication that the *vibratory energy of the magnetic elements in the molecules is not subject to the law of equipartition with the translational energy of the molecules*. The share which they possess is so small that the equivalent "temperature" approximates to the absolute zero. The internal field would require to be excessively *small* if we were to account for the observed values of the fractional residual magnetization on the postulate of equipartition at ordinary temperatures.

This question is of great importance in the theoretical treatment, and will be considered further afterwards (§§ 59, 67).

49. Weiss's Experimental Results.—The first experiments of Weiss, who was the pioneer in this work, were made on crystals of magnetite, which, like nickel, crystallizes on the cubic system with the body diagonals giving the directions of stable magnetization under the action of the internal field. His diagrams serve

well in indication of the peculiar nature of the magnetization of cubic crystals of this class. He first showed that a cubic crystal is *magnetically* non-isotropic.

The general features of the trend of magnetization, and also of residual magnetization in various fields, acting in the directions of the principal axes, are exhibited in Fig. 16. Some peculiarities exist which are, along with those in his other diagrams

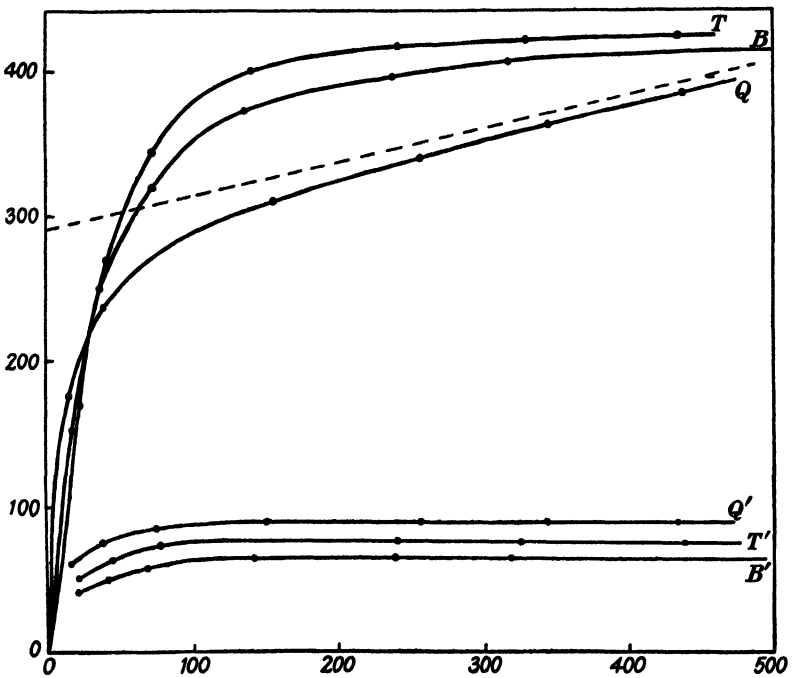

FIG. 16.

referred by Weiss, without doubt correctly, to the irregularities of crystalline construction. The reversal of the order of the ternary (T), binary (B), and quaternary (Q), magnetizations in weak fields is one such. And the relative reversal with regard to residual magnetization (T', B', Q') is another. In Figs. 17, 18 he showed the variation of the components of magnetization parallel and perpendicular to the external field; the former refers to magnetization in a face plane of the cube and the latter to magnetization in a binary plane. In the quaternary plane, the longitudinal component is four-lobed, and the transverse component

is eight-lobed. Theoretically these should be quite symmetrical axially. The variations make evident the structural crystalline defects to which Weiss referred. In the binary plane, the longitudinal component is two-lobed and the transverse component is eight-lobed—very unequally in sets of four. The curves have not been completed in all four quadrants. Quadrantal symmetry should be evident in a perfect crystal: but the difference between

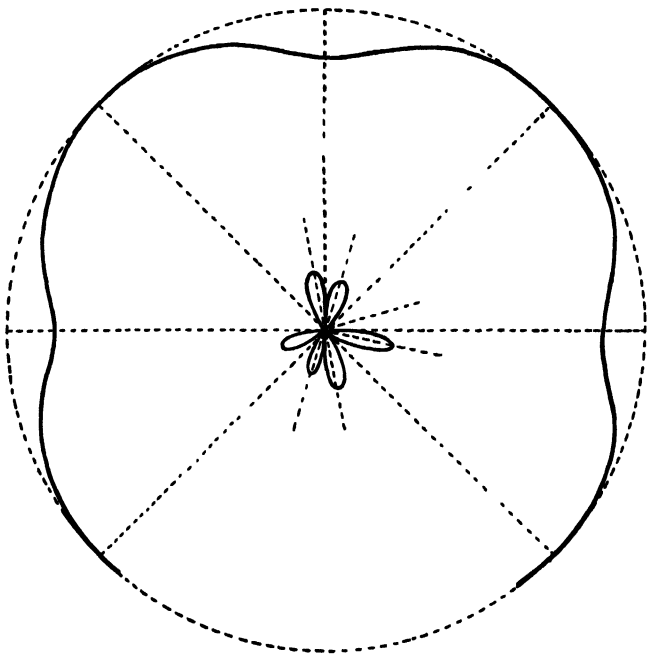

FIG. 17.

the two sets of four transverse lobes in the *binary* plane is fundamental, and is due to the obliquity of the two binary axes which lie in that plane.

Markedly peculiar results were obtained in the case of crystals of pyrrhotite, which crystallizes in a quasi-hexagonal prismatic form. The phenomena of magnetization under forces parallel to the hexagonal base are repeated within directions inclined successively at angles of 60° to each other. Weiss gave reasons for the supposition that the apparent hexagons were constituted of three juxtaposed rhombohedra; and he showed that, in one direction

in each elementary rhombohedron, magnetization occurred readily, while in a direction at right angles to it, the magnetization was much less easily effected. And the action was still less easy of accomplishment in a direction parallel to the edges of the prisms. He therefore spoke of a plane parallel to the bases of the prisms as the magnetic plane. In that plane the ratio of the principal apparent resistances to magnetization were nearly

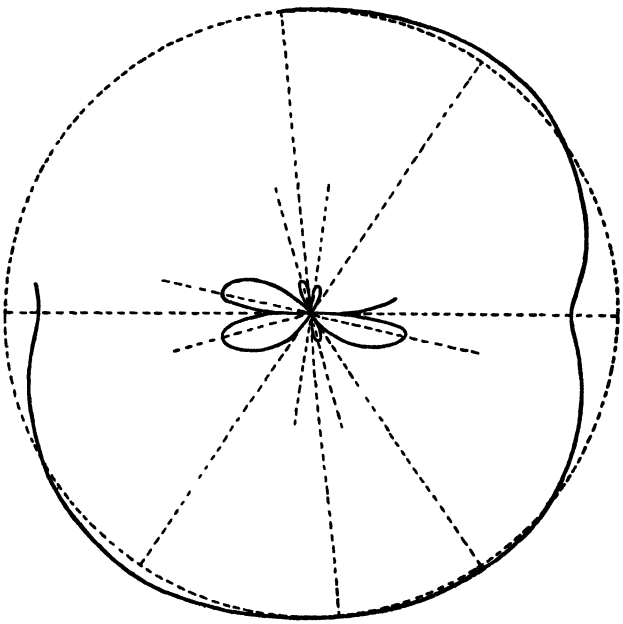

FIG. 18.

500 : 1. Perpendicularly to it, the resistance was about twenty times larger still than the greater of these.

These examples of experimental results regarding crystals will serve sufficiently in the matter of illustrative test of theory.

50. Weiss's Theory of Magnetization in Crystals.— In § 43 we have already considered Weiss's application of Langevin's discussion of the effect of temperature on the magnetization of an entirely random set of molecular magnets exerting no mutual influence. He considered a similar magnetic medium, but introduced the condition of sufficient mutual nearness of

the molecules to create an internal field, which he presumed to be proportional to the intensity of magnetization and to oppose the external field. His discussion constituted a thermomagnetic theory of paramagnetism, and was found to have applications to some cases of even the solid state. At a much earlier period, in fact contemporaneously with his experimental work on magnetic crystals, he gave his non-thermal theory of the action in crystals.

Ewing's contribution to the theoretical development of the subject was of the nature of a direct application of Weber's theory in so far as deductions could be made from it without special mathematical formulation. Weiss also worked without direct development of the requisite mathematical expressions involved in Weber's theory. But, like Weber and Maxwell, he introduced simple hypothetical expressions for the quantities dealt with. And, as Weber gave to the simple expression which he adopted a form designed to apply to the case of magnetization which he had specially in view (the magnetization of soft iron): also as Maxwell gave, to the simple extension of Weber's expression which he adopted, a form designed to apply to the case of magnetization which he had specially in view (the magnetization of hard iron or steel): so did Weiss give, to the simple expressions which he adopted, a form designed to apply to the case of magnetization which he had specially in view (the magnetization of a crystal). In each of these three cases, the expressions postulated for purposes of trial were expressions for the resultant effect as influenced by mutual magnetic molecular actions.

Weiss assumed that, in the case of pyrrhotite, the direction of easy magnetization was one in which a resultant internal field acted in the direction of magnetization, so that "spontaneous" quasi-saturation of the crystal could be maintained in that direction. In transverse directions the molecular field opposed magnetization. He further postulated that, in each principal direction, the field was proportional to the magnetization. Thus in pyrrhotite N_1, N_2, N_3 being the principal constants, the numerical values obtained gave $I(N_1 - N_2) = 73{,}000$, $I(N_1 - N_3) = 150{,}000$ gauss. These fields, extremely high if of magnetic origin, were necessary in order to express the excessive resistances to magnetization in any but the one direction of easy magnetization, in which quasi-saturation was effected by a field of 15 gauss. If it had to be admitted that the origin was other than magnetic

in its nature, the simplicity of Weber's postulate would disappear. The question is the very fundamentally important one which is to be considered more fully later (§§ 59, 67).

With regard to iron, Weiss presumed that iron crystals had properties similar to those of the elementary crystals in pyrrhotite. With random orientation, the spontaneous quasi-saturation of individual crystals produces no resultant magnetization until an external field is applied, when magnetization proceeds normally. For the spontaneous magnetization of any crystal, if it were directed on the whole oppositely to the external field, would be suddenly reversed when the component of the external field acting on it just overcame the coercive force which accompanied the "spontaneous" condition. In this way a rapid mounting of magnetization would follow, and ultimately, under the given conditions of temperature and the constants of the molecular field, absolute saturation would be gradually approximated to, as the magnitude of the external field rose.

That interaction of the spontaneous magnetization and the external field also gave a ready general explanation of the existence and nature of the hysteresis loop which is exhibited under cyclic magnetization by a cyclically varying external field. Nevertheless, in so far as the postulate of the existence of iron crystals, having properties resembling those of pyrrhotite, is concerned, the explanation requires change; for it is now known that crystals of iron are cubic, and that the *crystalline* lattice is either body-centred or face-centred.

In fact, Weiss's postulate of principal molecular fields proportional to the intensities of magnetization in their directions fails, in the case of cubic crystals, to give an approximate representation of Weber's internal field due to mutual interactions of magnetic molecules. It would lead to complete isotropy of magnetic phenomena in these crystals. For the three principal molecular field constants must, by symmetry, be equal; and therefore the field and the magnetization must be co-directed. It follows that the crystalline quality which appears in the magnetization of cubic crystals must either be introduced specially into the postulates regarding the form of the law which will approximate sufficiently to that which is compelled by Weber's theory, or the results of that theory must be worked out in detail without postulate.

This evaluation was effected soon after Weiss's earlier work was published. Discussion of it is given in the following chapter.

51. The work of Honda and Okubo, Frivold, and Webster.—Honda and Okubo worked out the law of magnetization of a system of randomly orientated cubic crystals on the simplifying postulate that all the crystals had one cube edge lying in a common direction, to which the external field was perpendicular. The problem was thus reduced to one involving a single variable instead of two; and the result was in close agreement with the experimental observations. They extended the postulate regarding the form of the law of action, given in Weber's treatment, with respect to the difference in direction of the magnetization and the external field. And they did so in a way which does not differ greatly from the directly deduced law, which is really the simplest law consistent with the cubic symmetry.

W. T. Webster, and, earlier, K. Beck, investigated the properties of crystals of iron. Some of their diagrams are shown in Chapters VII, VIII, Figs. 26–9, in connection with the verification of theory. Webster's empirical expression, assumed in order to get close agreement with observation for the component of the internal field perpendicular to the direction of magnetization, is worthy of note both from its efficiency and also because of its near correspondence with the expression deduced directly from Weber's theory.

The work of Frivold, to be discussed later also (§ 81), proceeded by means of the direct evaluation of the internal Weberian field. In connection with it and also with some of the other work alluded to above, the question of the share of mechanical molecular energy taken by the magnetic elements is very important.

CHAPTER VI

THE INTERNAL FIELD IN A HOMOGENEOUS CRYSTALLINE MEDIUM

"Though we have made use of terms having a physical meaning, such as electrical energy, each of these terms is regarded not as a physical phenomenon to be investigated, but as a definite mathematical expression. A mathematician has as much right to make use of these as of any other mathematical functions which he may find useful, and a physicist, when he has to follow a mathematical calculation, will understand it all the better if each of the steps of the calculation admits of a physical interpretation."—MAXWELL.

52. The Potential of a Magnet at a Distant Point.—In § 9 we have already found an approximate expression for this quantity. The approximation was made by considering the difference of the distances of the two poles from an external point to be equal to the projection of the length of the magnet upon the line drawn from the centre of the magnet to that point. We must now obtain a more accurate expression: for it is upon the closeness of the approximation that the effect of a homogeneous crystalline arrangement of magnets depends, and that most expressly in the case of cubic crystals, which are exemplified in the structure of the more strongly magnetic metals.

Before entering upon the investigation, it is well to realize fully the weight of the remark of Maxwell, quoted above, regarding the employment of mathematical expressions which have a physical interpretation in the discussion of problems in mathematical physics. The remark occurs in the course of his treatment of our present problem amongst others. And it is worthy of note that, as not infrequently occurs in the interrelations of mathematics and physics, Maxwell in that treatment not merely gave a physical discussion superior to the usual one, but also introduced an improvement in the pure mathematics which was employed normally. The discussion can be followed readily by anyone who

has a knowledge of the merest elements of the differential calculus; and that is a qualification possessed now by most students, who are aiming at Higher University work in such subjects, even before they leave the public schools.

A quantity m of north magnetism placed at a point C (Fig. 19) produces at any other point P the potential $-m/r$, where r is the distance of P from C. For, potential being defined as the work performed by the forces of repulsion on a unit of north magnetism, it is identical with the potential energy of the unit relatively to m, and so decreases as r increases. Thus, since it is zero at infinity, it must be taken as negative at finite distances. Now let m be displaced to the small distance a from C in a direction which is also indicated by a in the same way as magnitude and direction are usually indicated by the radius vector r: and let μ be the cosine of the angle between the lines PC and a.

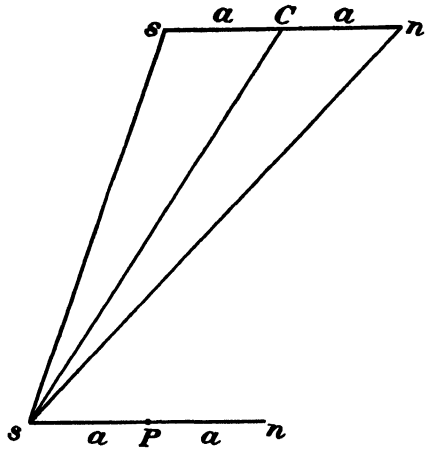

FIG. 19.

We thus have

$$\frac{\partial r}{\partial a} = \mu = \frac{p}{r}$$

where p is the component of r in the direction of a: and also

$$\frac{\partial \mu}{\partial a} = \frac{r\dfrac{\partial p}{\partial a} - p\dfrac{\partial r}{\partial a}}{r^2} = \frac{1}{r}(1 - \mu^2)$$

for $\partial p/\partial a = 1$, since the directions of p and a are the same. The potential $-m/r$ at P due to m near C now becomes

$$-\frac{m}{r} - \frac{\partial}{\partial a}\left(\frac{m}{r}\right)a - \frac{1}{2}\frac{\partial^2}{\partial a^2}\left(\frac{m}{r}\right)a^2 - \ldots - \frac{1}{n!}\frac{\partial^n}{\partial a^n}\left(\frac{m}{r}\right)a^n - \ldots$$

Similarly, if $-m$ be placed at the point $-a$ from C, the potential at P becomes

$$+\frac{m}{r}-\frac{\partial}{\partial a}\left(\frac{m}{r}\right)a+\frac{1}{2}\frac{\partial^2}{\partial a^2}\left(\frac{m}{r}\right)a^2-\ldots(-)^n\frac{1}{n!}\frac{\partial^n}{\partial a^n}\left(\frac{m}{r}\right)a^n+\ldots$$

But this arrangement constitutes a magnet, whose moment is $M = 2am$, and whose direction is that of a. So the potential produced at P by that magnet is the sum of these quantities, and therefore is

$$V=-M\left[\frac{\partial}{\partial a}\left(\frac{1}{r}\right)+\frac{1}{3!}\frac{\partial^3}{\partial a^3}\left(\frac{1}{r}\right)a^2+\ldots+\frac{1}{n!}\frac{\partial^n}{\partial a^n}\left(\frac{1}{r}\right)a^{n-1}+\ldots\right]$$

where n is odd.

53. The Mutual Potential Energy of Two Similar Small Magnets Similarly Oriented.—If we now place an amount of north magnetism m at P the potential energy in the system formed by the magnet at C and the quantity of north magnetism, m, at P is mV. If now we displace m from P to the distance a in the direction a, the energy becomes

$$m\left[V+a\frac{\partial}{\partial a}V+\frac{a^2}{2}\frac{\partial^2}{\partial a^2}V+\ldots+\frac{a^m}{m!}\frac{\partial^m}{\partial a^m}V+\ldots\right]$$

And, if we then place $-m$ at $-a$ the added energy is

$$-m\left[V-a\frac{\partial}{\partial a}V+\frac{a^2}{2}\frac{\partial^2}{\partial a^2}V-\ldots(-)^m\frac{a^m}{m!}\frac{\partial^m}{\partial a^m}V+\ldots\right]$$

Therefore the mutual potential energy of the two parallel similar magnets formed in this way is, with m odd,

$$M\left[\frac{\partial}{\partial a}+\frac{a^2}{3!}\frac{\partial^3}{\partial a^3}+\frac{a^4}{5!}\frac{\partial^5}{\partial a^5}+\ldots(-)^{m-1}\frac{a^{m-1}}{m!}\frac{\partial^m}{\partial a^m}+\ldots\right]V$$

$$=-M^2\left[\frac{\partial}{\partial a}+\frac{a^2}{3!}\frac{\partial^3}{\partial a^3}+\frac{a^4}{5!}\frac{\partial^5}{\partial a^5}+\ldots\right]\left[\frac{\partial}{\partial a}+\frac{a^2}{3!}\frac{\partial^3}{\partial a^3}+\frac{a^4}{5!}\frac{\partial^5}{\partial a^5}+\ldots\right]\frac{1}{r}$$

$$=-M^2\left[\frac{\partial^2}{\partial a^2}+2\frac{a^2}{3!}\frac{\partial^4}{\partial a^4}+\left(2\frac{a^4}{5!}+\frac{a^4}{3!\,3!}\right)\frac{\partial^6}{\partial a^6}+\ldots\right]\frac{1}{r}$$

It is unnecessary for our present purposes to carry the series beyond the second term. But, in the case of a crystal belonging to the cubic system, the second term is the most important one, for the first, when summed over a large number of magnets situated on a cubic lattice, is zero, as we shall see.

We have now to evaluate the second and fourth order differentials in the formula. For this purpose the two relations already given,

$$\frac{\partial r}{\partial a} = \mu = \frac{p}{r}, \quad \frac{\partial \mu}{\partial a} = \frac{1 - \mu^2}{r}$$

are sufficient. They give

$$\frac{\partial^2}{\partial a^2}\left(\frac{1}{r}\right) = \frac{1}{r^3}(3\mu^2 - 1)$$

and

$$\frac{\partial^4}{\partial a^4}\left(\frac{1}{r}\right) = \frac{3}{r^5}(35\mu^2 - 30\mu + 3)$$

If we write

$$3\mu^2 - 1 = 2P_2, \qquad 35\mu^2 - 30\mu + 3 = 8P_4$$

the mutual potential energy becomes

$$-\frac{2M^2}{r^3}\left[P_2 + \left(\frac{a}{r}\right)^2 2^2 . P_4 + \ldots + \left(\frac{a}{r}\right)^{2(n-1)} 2^{2(n-1)} P_{2n} + \ldots\right]$$

where the *Ps* are functions of μ, the cosine of the angle between the line of centres and the common direction of the two magnets. The mathematical function $P_{2n}(\mu)$ is known as the *Zonal Harmonic* of degree $2n$ in μ. Thus Maxwell's simple treatment has resulted in the development of the mutual potential energy in a series of zonal harmonics. The terms diminish very rapidly as n increases if the size of the magnets is small relatively to their distance apart.

By prefixing the sign of summation to the above expression, we extend it to the case of the mutual potential energy, V_0,

between the magnet at the origin and the totality of its surrounding magnets, and so have

$$V_0 = -\sum \cdot \frac{2M^2}{r^3}\left[P_2 + \left(\frac{a}{r}\right)^2 2^2 P_4 + \ldots + \left(\frac{a}{r}\right)^{2(n-1)} 2^{2(n-1)} P_{2n} + \ldots\right]$$

Similarly, we can express by a double summation, the total magnetic potential energy of the whole system as

$$V_T = -\sum\sum \cdot 2\frac{M^2}{r^3}\left[P_2 + \left(\frac{a}{r}\right)^2 2^2 P_4 + \ldots + \left(\frac{a}{r}\right)^{2(n-1)} 2^{2(n-1)} P_{2n} + \ldots\right]$$

the factor 2 being removed because each pair of magnets must be taken once only.

54. Potential of the Internal Field (continued).—It has already been remarked that P_2 vanishes in the case of a cubic crystal, which alone we are to consider in detail. So we take

$$V_0 = -\sum \cdot \frac{2M^2}{r^3}\left(\frac{a}{r}\right)^2 \frac{1}{2}(35\mu^4 - 30\mu^2 + 3)$$

The reason for the vanishing of P_2 is that, because of the positive and negative symmetry of the *rs* measured from the centre of any magnet in the system, the average inclination of r to the three axes of the cubic structure is zero. So the average values of the squares of the cosines of the inclinations of r to these axes are equal. But this condition is satisfied by a body diagonal of the cube, which is equally inclined to the three edges, and the cosine of its inclination is $\mu = 1/\sqrt{3}$. Hence $3\mu^2 = 1$.

The quantities r and μ have now to be expressed in terms of other quantities by means of which the three summations can be

carried out. Since, in any crystalline arrangement of molecules, if we join any two centres which are nearest to each other in any direction, other centres are to be found at successive equal distances on either side of these two (Fig. 20), so, if we write $r^2 = x^2 + y^2 + z^2$, where x, y, and z are measured parallel to the edges of the cubic structure from the origin at which V_0 is to be found; and, if ρ be the least distance between planes parallel

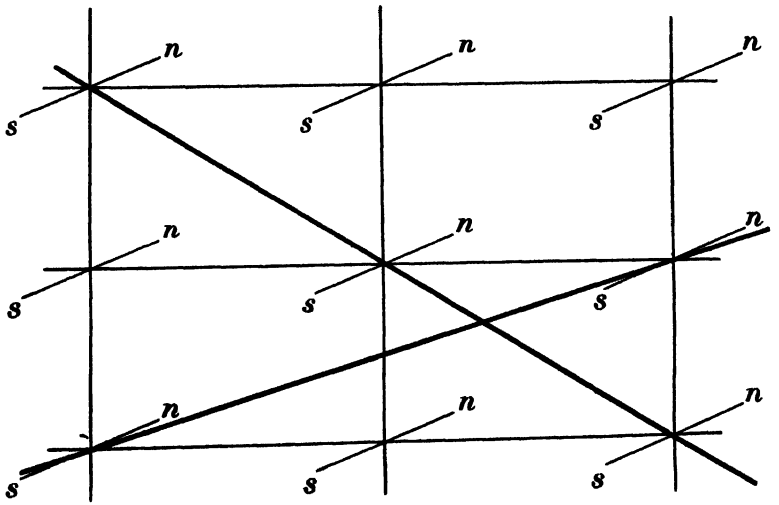

FIG. 20.

to the faces of the cubic structure on which centres of molecules lie, we have

$$x = l\rho, \quad y = m\rho, \quad z = n\rho$$

where l, m, n are positive or negative integers.

And, if α, β, γ be the direction cosines of the axes of the molecular magnets, while the direction cosines of r are x/r, y/r, z/r, we have

$$\mu = \frac{\alpha x + \beta y + \gamma z}{r}$$

$$= \frac{\alpha l + \beta m + \gamma n}{(l^2 + m^2 + n^2)^{1/2}}$$

$$V_0 = -\frac{M^2a^2}{\rho^5}$$
$$\sum\left[35\frac{(\alpha^2l^2+\beta^2m^2+\gamma^2n^2+2\alpha\beta lm+2\beta\gamma mn+2\gamma\alpha nl)^2}{(l^2+m^2+n^2)^{9/2}}\right.$$
$$\left.-30\frac{(\alpha^2l^2+\beta^2m^2+\gamma^2n^2)}{(l^2+m^2+n^2)^{7/2}}+3\frac{1}{(l^2+m^2+n^2)^{5/2}}\right]$$

the products of lm, mn, nl vanishing on summation since positive and negative values occur equally. Also, by symmetry,

$$\sum\frac{l^4}{(l^2+m^2+n^2)^{9/2}}=\sum\frac{m^4}{(l^2+m^2+n^2)^{9/2}}$$
$$=\sum\frac{n^4}{(l^2+m^2+n^2)^{9/2}}=\frac{D}{35}$$

$$\sum\frac{3l^2m^2}{(l^2+m^2+n^2)^{9/2}}=\sum\frac{3m^2n^2}{(l^2+m^2+n^2)^{9/2}}$$
$$=\sum\frac{3n^2l^2}{(l^2+m^2+n^2)^{9/2}}=\frac{G}{35}$$

$$\sum\frac{l^2}{(l^2+m^2+n^2)^{7/2}}=\sum\frac{m^2}{(l^2+m^2+n^2)^{7/2}}$$
$$=\sum\frac{n^2}{(l^2+m^2+n^2)^{7/2}}=\frac{A}{30}$$

$$\sum\cdot\frac{3}{(l^2+m^2+n^2)^{5/2}}=\mathcal{J}$$

Therefore, since $\alpha^2+\beta^2+\gamma^2=1$,

$$V_0=-\frac{M^2a^2}{\rho^5}[D(\alpha^4+\beta^4+\gamma^4)+2G(\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2)-A+\mathcal{J}]$$
$$=-\frac{Ma^2}{\rho^5}[(D-G)(\alpha^4+\beta^4+\gamma^4)+G-A+\mathcal{J}]$$
$$=-\frac{Ma^2}{\rho^5}[\tfrac{3}{5}-(\alpha^4+\beta^4+\gamma^4)](G-D)$$

for, obviously, $3A=10\mathcal{J}$, and $6G+9D=35\mathcal{J}$.

The summations of G and D are readily carried out for small values of the integers l, m, n. The magnitude and sign of $(G - D)$ depends on the nature of the cubic lattice, of which there are three possible. These are the simple cubic lattice, in which molecules are centred at cube corners, so that each has six nearest neighbours; the body-centred lattice, in which the cube has also a molecule situated at its centre, so that each molecule has eight nearest neighbours; and the face-centred cubic lattice in which molecules are placed at face centres as well as at the corners, so that each has twelve nearest. The latter is of greatest interest physically, as it presents the grouping which would most naturally occur under attractive forces. The two latter are known to exist in the case of the strongly magnetic metals.

55. The Internal Field.—If the crystalline body be finite, and be placed in an external field sufficiently strong to give average co-direction of the molecular magnets at the temperature of observation, the ensuing surface distribution of magnetization gives rise to a component of the internal field, which may be found by the usual treatment as employed in connection with surface distributions of electricity.

Similarly, if we consider a sphere, surrounding the magnet at the origin, of radius R large in comparison with the distance between adjacent molecules, and having the molecules in its interior removed, the distribution of the intensity of magnetization on the interior of the sphere enables us to show that the internal field due to all the rest of the system of molecules, supposed to extend to infinity, is

$$-\frac{4}{3}\pi I$$

This is a *uniform* field. Any non-uniform field which may exist as a consequence of the molecular structure is due to the molecules within the range R. Indeed the evaluation of the first two terms in the expression for V_0 shows that the variable part of the field exerted on a molecule by its neighbours is practically accounted for by consideration only of those which are within a distance equal to a few least molecular distances: so that the calculation of its value numerically is comparatively easy.

Now in a homogeneous crystalline aggregate the various mole-

cular distances are fixed, so that $\mu(=\cos\theta)$, or θ, is the only variable. Consequently $-\partial V_0/\partial\theta = T$ is the torque exerted, equally on each magnet, to cause orientation of the direction of magnetization. If we regard a as variable, we can similarly find the value of $-\partial V_0/\partial a = L$, the longitudinal component of the internal field, that is, the component parallel to the direction of magnetization. From symmetry we see that the mechanical force on a north pole is balanced by that on the south pole of the same magnet, so that no change in any r results.

56. Force Components in the Field.—The components of the internal forces which are of importance in the present problem are those parallel and perpendicular, respectively, to the direction of magnetization, i.e. to the direction of a in the preceding work. So, force being the space rate of variation of energy, the internal force tending to alter the direction of magnetization is given by

$$F = -\frac{\partial V_0}{2a\partial\theta} = -\frac{1}{2a}\frac{\partial V_0}{\partial\mu}\cdot\frac{\partial\mu}{\partial\theta}$$

$$V_0 = -\sum\frac{M^2a^2}{r^5}(35\mu^4 - 30\mu^2 + 3)$$

Therefore, T being the strength of the field transverse to a (consequently measured per unit quantity of magnetism), we get

$$\frac{F}{2m} = T = \sum\cdot\frac{Ma^2}{r^5}10(7\mu^2 - 3)\mu\sqrt{1-\mu^2}$$

The presence of the square root prevents ready evaluation of T, but the components of force, T_l, T_m, T_n, parallel to the cube edges, can be calculated easily. To express these components we use

$$\begin{aligned}1-\mu^2 &= 1-(al+\beta m+\gamma n)^2\\ &= [l-a(al+\beta m+\gamma n)]^2 + [m-\beta(al+\beta m+\gamma n)]^2\\ &\qquad + [n-\gamma(al+\beta m+\gamma n)]^2\end{aligned}$$

Therefore

$$\begin{aligned}T_l = 10\sum\cdot\frac{M^2a^2}{r^5}&[l-a(al+\beta m+\gamma n)]\\ &(al+\beta m+\gamma n)[7(al+\beta m+\gamma n)^2-3]\end{aligned}$$

the other two component torques being obtained from this expression by cyclical interchange of l, m, n and α, β, γ. Then, remembering that any term which involves an odd power of l, m, or n cancels out on summation, we obtain

$$T_l = 2\frac{Ma^2}{\rho^5}(G - D)\alpha[(\alpha^4 + \beta^4 + \gamma^4) - \alpha^2]$$

$$T_m = 2\frac{Ma^2}{\rho^5}(G - D)\beta[(\alpha^4 + \beta^4 + \gamma^4) - \beta^2]$$

$$T_n = 2\frac{Ma^2}{\rho^5}(G - D)\gamma[(\alpha^4 + \beta^4 + \gamma^4) - \gamma^2]$$

The components of torque can be written down at once since we have

$$\Theta_l = (T_n\beta - T_m\gamma)M = 2\frac{M^2a^2}{\rho^5}(G - D)\beta\gamma(\beta^2 - \gamma^2)$$

Hence we can put the square of the resultant torque

$$\Theta^2 = \Theta_l^2 + \Theta_m^2 + \Theta_n^2$$

in the symmetrical form

$$4\frac{M^4a^4}{\rho^{10}}(G - D)^2[(\alpha^6 + \beta^6 + \gamma^6) - (\alpha^4 + \beta^4 + \gamma^4)^2]$$

Lastly, to find the value of the longitudinal component of the internal field, we note that it tends to increase the length of the magnet through its action on m and $-m$. Therefore the magnitude of the compcnent field strength is

$$L = -\frac{1}{m}\frac{\partial V_0}{2\partial a} = \sum\frac{2M^2a^2}{m2ar^5}(35\mu^4 - 30\mu^2 + 3)$$

$$= \sum\cdot 2\frac{Ma^2}{r^5}(35\mu^4 - 30\mu^2 + 3)$$

Replacing in this expression $\mu = (\alpha l + \beta m + \gamma n)(l^2 + m^2 + n^2)^{-1/2}$, and again remembering that no term which contains l, m, or n raised to an odd power can appear in the summation, we get

$$L = 2\frac{Ma^2}{\rho^5}(G - D)\left[\tfrac{3}{5} - (\alpha^4 + \beta^4 + \gamma^4)\right]$$

57. Special Cases of the Force Components.—The most important sections of the surfaces represented by these expressions are those with, say, $\gamma = 0$, corresponding to magnetization in a face plane of the cube; $\alpha = \beta$, corresponding to a binary plane; and $\alpha + \beta + \gamma = 0$, corresponding to a ternary

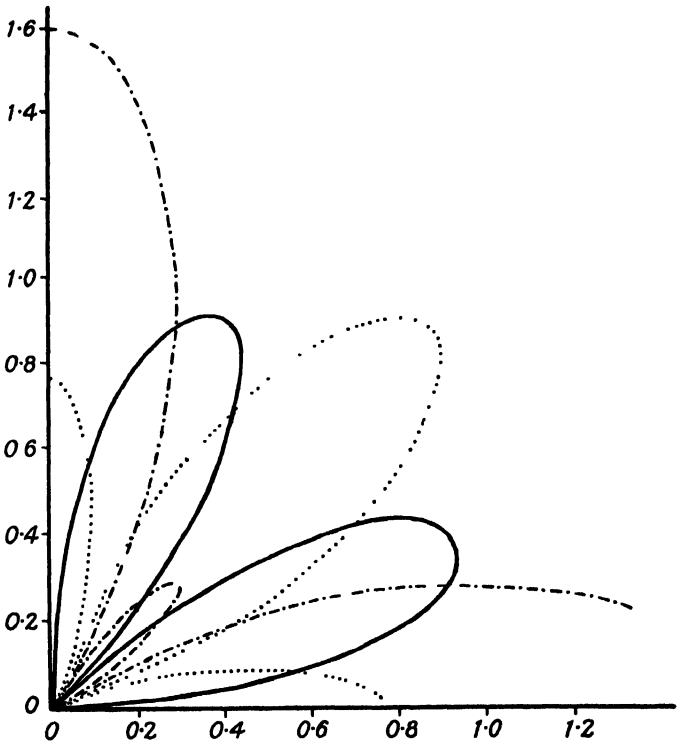

FIG. 21.

plane, i.e. one perpendicular to a body diagonal of the cubic structure.

In Fig. 21 the two lobes drawn in continuous lines represent the transverse component on an arbitrary scale, in the case of magnetization in a face plane. The radii represent the magnitude of the transverse component of the internal force acting on a pole when the magnetization is in the direction of the radius. The axes in the diagram are parallel to cube edges. The transverse component vanishes, when the magnetizatien is along a

cube edge or along a binary axis (45°). Its direction reverses after passing through zero; that is to say, alternate lobes represent oppositely orientated transverse fields. Thus the internal transverse field is either directed towards a quaternary axis (cube edge), or away from it. The actual condition is determined by the cubic structure, and so by the sign of $(G - D)$. In the closest packed system (face-centred cube) G is greater than D, and so the force is directed from the quaternary axis. Therefore magnetization along a quaternary axis is unstable, while that along a binary axis (face diagonal) is stable, so far as magnetization in a face plane is concerned. The same is true with regard to the next closest packed arrangement (body-centred cube); but the opposite statement holds in the case of the most open packed arrangement, for in it the summations give $D > G$.

In a binary plane which contains a quaternary axis, a binary axis, and two ternary axes (cube diagonals), two lobes also occur in the quadrant; but, since the angle between two cube diagonals is not a right angle, one lobe is small and the other large, the largest lobe being adjacent to the quaternary axis. Here the transverse force is directed towards the ternary axes, and therefore from the quaternary axes and also from the binary axes, if $G > D$. That is to say, in the closest packed, and the next closest packed, cubic arrangement of molecular magnets, magnetization parallel to the ternary axes, i.e. parallel to the body diagonals of the cube, is in equilibrium; but, in the simple cubic system (most open packing) the stable directions of magnetization are parallel to the cube edges.

The dot-dash curve in Fig. 21 represents the variable part of the longitudinal component of the internal field for magnetization in the face plane. The summation has been carried to values of l, m, n up to 10. The additional field, $4\pi I/3$ approximately, as already stated, has to be added to take account of the remainder. To evaluate the sums in the face-centred system, it is necessary, with the origin at a corner, P, where a magnet is placed, to make ρ be half of a cube edge. Then, if l is even, m and n are both even or both odd; and if l is odd, m is even or odd according as n is odd or even. The dotted curve in Fig. 21 shows, on a much larger scale, the variable part of the longitudinal component of force, in the face plane, due to the next zonal harmonic which is neglected in the preceding work.

The expression for L at the end of the preceding section shows that the longitudinal component of the internal field is negative in the face-centred system in the neighbourhood of the cube edges in a face plane; while, in the neighbourhood of the binary axis, it is slightly positive.

In a plane perpendicular to a ternary axis (1, 1, 1) the longitudinal component of the internal field is constant; and, as

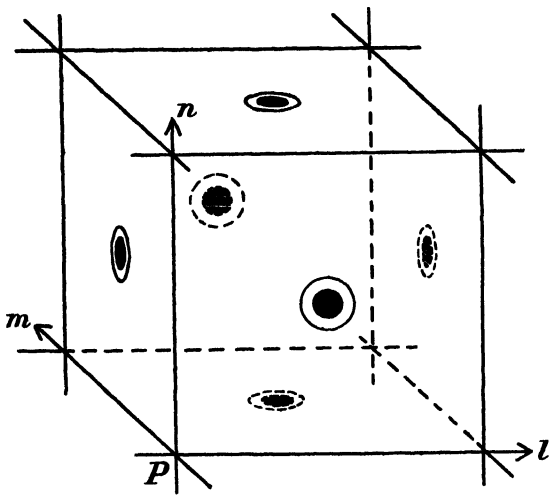

FIG. 22.

follows from the crystalline symmetry, the transverse component is positive and negative alternatively as a plane passing through the line (1, 1, 1) and the direction of magnetization revolves through successive angles of 60°. It vanishes when that plane passes through a quaternary or a binary axis.

For further details, and for the treatment of the field in crystals of the various non-cubic forms, reference may be made to papers in the *Proc. R.S. Edin.*, 1905, 1908, 1912, and, specially, in the *Trans. R.S.E.*, 1926.

CHAPTER VII

THE MAGNETIZATION OF A HOMOGENEOUS CRYSTALLINE MEDIUM

58. The Intensity and its Components.—The case of magnetization in a face plane of a cubic crystal will alone be dealt with in detail. The work of the preceding chapter has shown that, when a crystal is magnetized steadily in any direction, a transverse component of the internal field comes into play,

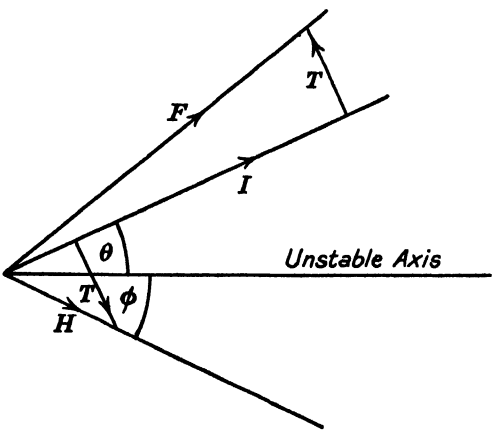

FIG. 23.

and has to be annulled by a component of the external field H. The longitudinal component of that field may have many values depending on the magnitude of H. And the intensity itself may have many values dependent on the values of the total longitudinal field, on the temperature, and, of course, on the direction of magnetization.

If the magnetization I be in a direction inclined at an angle θ to the axis of instability (Fig. 23) the internal field F has a component T perpendicular to the direction (α, β, γ) of magnetization,

acting so as to increase θ. And this component is balanced by $H \sin \psi$, where $\psi = \theta + \phi$ is the angle between I, (α, β, γ) and H, (l, m, n), ϕ being the angle which H makes (negatively) with the direction of instability. Under these conditions, the total field in the direction of magnetization is $H \cos \psi + L$, where L is the longitudinal component of the internal field, taken as positive when co-directed with I. And, apart from special data known regarding the nature of the "molecular magnet," we have no information regarding I other than that expressed by the condition

$$I = f(H \cos \psi + L)$$

where f is an unknown function. Final appeal regarding its form can only be made to experiment. But the simplest assumption which we can make for purposes of trial is that f indicates direct proportionality at any fixed temperature. Thus

$$I = c_0(H \cos \psi + L)$$

$$T = H \sin \psi$$

where c_0 has to be determined so as to give a finite saturation value to I, and the heat motions are at present neglected.

Replacing for T and L from § 56, with the substitutions

$$A = \frac{Ma^2(G - D)}{2\rho^5}$$

$$\alpha = \cos \theta, \quad \beta = \sin \theta, \quad \gamma = 0$$

we get, from $I_t = I \sin \psi$, $I_l = I \cos \psi$,

$$I_t = c_0 A \left[\sqrt{1 - \frac{A^2}{H^2} \sin^2 4\theta} - \frac{A}{H}\left(\frac{3}{5} + \cos 4\theta\right) \right] \sin 4\theta$$

$$I = c_0 H \left[\sqrt{1 - \frac{A^2}{H^2} \sin^2 4\theta} - \frac{A}{H}\left(\frac{3}{5} + \cos 4\theta\right) \right] \sqrt{1 - \frac{A^2}{H^2} \sin^2 4\theta}$$

as the components of magnetization transverse to, and parallel to, the direction of the imposed field H.

From the expressions involving ψ we have

$$\frac{I_t}{I_l} = \frac{A \sin 4\theta}{\sqrt{H^2 - A^2 \sin^2 4\theta}} = \frac{T}{L}$$

That is to say, the component of magnetization transverse to the external field bears to the component of the field transverse to the magnetization the same ratio as the component of magnetization parallel to the field bears to the component of the field parallel to the magnetization.

59. The Magnitude of the Internal Field.—The square of the internal field, being

$$F^2 = A^2 \sin^2 4\theta + A^2(\tfrac{3}{5} + \cos 4\theta)^2$$

has its stationary values where $\sin 4\theta = 0$, and so at $\theta = 0$ and $\theta = \pi/4$. Thus the maximum value of the internal field, equal to the coercive field in quasi-saturation, has the value $8A/5$, and its minimum value is $2A/5$. But if a loop of transverse magnetization is completed ($\theta = 0$ to $\theta = \pi/4$) $\sin 4\theta$ must take all values from 0 to 1. So the occurrence of the expression under the sign of the square root in the values of I_t and I_l show that H cannot be less than A. In other words *the maximum value of the internal field cannot be greater than eight-fifths of the smallest external field which is sufficient to enable the curves of the transverse and longitudinal components of magnetization to be experimentally completed.*

This conclusion is strongly at variance with the supposition that enormous internal fields can exist. And it is to be noted that the conclusion is *independent of any question regarding the nature, magnetic or otherwise, of the internal field itself.*

It is important therefore to explicitly state the postulates upon which the conclusion is reached. First, Weber's assumption of no influence other than that of the magnetic interactions of molecules is explicitly adopted. If phenomena should appear which necessitate the introduction of influence other than magnetic, Weber's assumption would require modification. Second, it is assumed that this magnetic influence may be ascribed to the existence of "molecular magnets." No specification is made regarding the

nature of these or their linkage with the molecules, beyond the condition that, apart from their magnetic interaction with others, they are freely orientatable—again Weber's view. Third, it is assumed that ideal magnets, representing these molecular magnets, are of such a length in relation to their average distance apart, that their effects on one of their number are representable in series, involving powers of the ratio of these lengths, and converging with sufficient rapidity to enable the most important term in the series to be used alone as a first approximation.

It is to be noted also that the effect of surface distribution is excluded by the presumption that the crystalline structure is homogeneous and infinite in extent. When boundaries are considered, the effects at an outside point, whose distance from the boundary is large in comparison with molecular distances, can be estimated by means of the surface distribution of intensity in the usual way. But the question is still open whether the equilibrium conditions are the same just within the surface as they are in the interior of the crystal. This can only be settled by an investigation of the same kind as those used in the work dealing with an infinite lattice, but in the summations involved in G and D (§ 65), the values of l, m, n referring to points outside the boundary have to be omitted.

These sums, G', D', say, are in closest packing, reversed as to magnitude in comparison with G, D. Whence it follows that the cube edges are, at a surface which is parallel to a cube face, the directions of intrinsic stability. This is of importance in connection with the effects of the presence of lamination in crystalline structure, and the possible explanation of the existence of residual magnetization, in directions in which the internal field opposes magnetization, such as Weiss's observations on magnetite seem to indicate.

60. Peculiarities of Magnetization in Magnetite.—The facts that (Fig. 16, § 49) the magnetizations observed in the higher fields parallel to the principal axes are found to be of increasing magnitude as we pass from the quaternary, through the binary, to the ternary axes, is in entire accordance with the theoretical view. For the crystals, being shown by X-ray methods to belong to the cubic system with a face-centred lattice, and the sum G in that lattice (§ 55) being greater than the sum D

(*Proc. R.S. Edin.*, 1905), the theory has shown that the quaternary axis should be one of unstable magnetization. The transverse component of the internal field acts so as to increase any deviation from alignment of the molecular magnets along that axis, and the parallel component is directed against the magnetization. In the face plane, the binary axis is the direction of stability, although that is only so in virtue of a thin plate of the substance being used. For the true direction of stability is that of a ternary axis, and the molecules are only prevented from setting their magnetic axes parallel to a ternary axis by the self-demagnetizing force due to the surface magnetization induced whenever they turn out of the face plane.

This enforced stability along a binary axis when the plate is thin merits careful consideration, for it shows that the internal forces, tending to compel alignment out of the plane of the face, have their efforts largely overcome by the force due to the surface magnetization which only produces effects of the magnitude of ordinary fields in the neighbourhood. Thus internal fields of ordinary magnitude are indicated.

But Weiss's curve Q mounts more slowly than either of the curves B and T in the middle part of its course. This main result is the natural consequence of the reverse internal field, which reduces the effective field considerably below the magnitude of the external field. The inversion of the order of the Q, B, T curves in weak fields, ascribed by Weiss, correctly no doubt, to crystalline flaws, may be due to binary and ternary defects.

Again, the inversion in the order of the residual curves Q', B', T', is opposite to what theory indicates. Because of the demagnetizing field in the quaternary direction, no magnetization should remain when the external field is withdrawn if the crystal were of infinite extent. Because of the thermal motions and the self-demagnetizing action of any groups of magnets which may become co-directed in the quaternary direction, residual magnetization must disappear. If, therefore, it does not disappear, but on the contrary is more evident in that direction than it is in the direction in which it would be expected, from theory, to be most evident, the point becomes one of almost crucial significance. The test of correctness of view is this, that, so far as the theory is concerned, the explanation of the contradiction connected with residual magnetization must follow as a natural

consequence of the view which is adopted for the explanation of peculiarities connected with the induced magnetization.

As said above, the increase of permeability as we pass from the quaternary, through the binary, to the ternary directions is predictable from theory. Its decrease in the same sequence at the lower field strengths cannot be predicted unless some structural change has taken place. But the fundamental crystalline structure is the same at the low and the high fields alike. Therefore we must look to the existence of flaws in the crystalline structure for one possible explanation.

Now no phenomenon is more well known regarding crystals than their tendency to lamination in definite planes. These are naturally the planes of closest packing of molecules, which, in the face-centred lattice of magnetite, are the ternary planes.

But the angle between the ternary and binary axes being smaller than that between the other two pairs, flaws perpendicular to the ternary axes will oppose magnetization most if it be directed along these lines, and least if it be directed along the cube edges. So the presence of these flaws might give rise to the inverted order of sequence for magnetization in weak fields. In stronger fields the effect of magnetostriction (§ 37) tends to bind together the walls of incipient fissures across which magnetization is proceeding. Thus the fissures become less efficient, and the natural sequence tells out.

When the strong fields are lessened for the purpose of testing the residual magnetization, the flaws against become effective, giving rise to what are effectively surface magnetizations on the boundaries of fissures and so producing demagnetizing fields which are most powerful in the ternary directions, and least so in the quaternary directions. All the peculiarities exhibited in Weiss's diagram are therefore direct consequence of the theory in its application to a crystal having a tendency to lamination in planes normal to the ternary directions. Further consideration of similar discrepancies will be made later (§§ 63, 66).

61. Characteristics of the Curves of Transverse Magnetization.—The flattening of these loops on the sides adjacent to the cube edges was pointed out by Weiss. The transverse component of the internal field which arises whenever the magnetization deviates from the edge directions helps the external

field to effect the orientation. At the same angular deviation of the magnetization from the binary direction, the transverse component of the internal field is, on the contrary, opposing the orientation.

Thus the transverse component of magnetization mounts rapidly with increasing deviation of the field from the edge

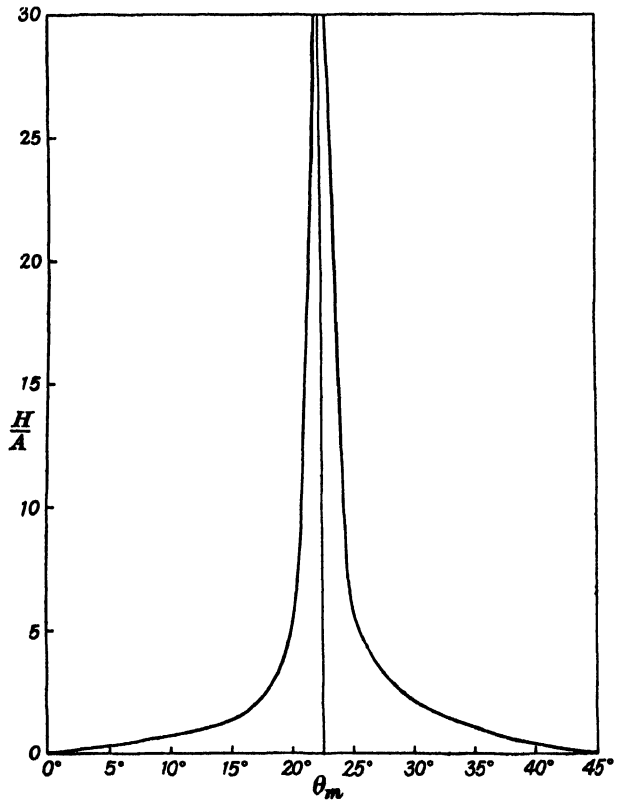

FIG. 24.

directions in the cube face, and more slowly with its increasing deviation from diagonal directions in the cube face.

By differentiation of the equation

$$I = c_0 A\left[\sqrt{1 - \frac{A^2}{H^2}\sin^2 4\theta} - \frac{A}{H}\left(\frac{3}{5} + \cos 4\theta\right)\right]\sin 4\theta$$

it is easy to obtain the condition for the maximum of I_t if we meanwhile regard c_0 as practically independent of θ. It is

$$\left(\frac{A}{H}\right)^4 [4\cos^2 4\theta \sin^2 4\theta + \{(\tfrac{3}{5} + \cos 4\theta)\cos 4\theta - \sin^2 4\theta\}^2] \sin^2 4\theta + \left(\frac{A}{H}\right)^2 [\text{same bracketed quantity}] + \cos^2 4\theta = 0$$

which can readily be shown to have real roots for $(A/H)^2$. So, A being presumed known for a crystal, this expression gives the value of the external field for which θ is the direction of magnetization corresponding to the maximum value of its transverse component. It is a universal expression, applicable to magnetization in a face plane of any cubic crystal, in terms of the "generalized" field, that is the external field expressed as a multiple of A, or of the maximum or minimum values of the internal field. The expression is represented by the curve in Fig. 24, H/A being the ordinate at which θ is the abscissa. Positive values only of the cosine are needed since an infinite external field is required to make the maximum of the transverse component of the magnetization occur at $\theta = \pi/8$. The cases with θ from $\pi/8$ to $\pi/4$ correspond to negative values of A, and are therefore really included. The flattening of the loops of transverse magnetization occurs then on the side adjacent to a binary axes.

62. Theoretical and Experimental Curves (continued).—The experimental observations are made in terms of the direction of the field, not of the magnetization. The equation connecting these is

$$\phi = \theta - \sin^{-1}\frac{T}{H} = \theta - \sin^{-1}\frac{A}{H}\sin 4\theta$$

where ϕ, measured like θ from the unstable axis, is the direction angle of the field.

The determination of θ in terms of the observed value of ϕ, with any value of A/H, may be found by replacing for A/H, from the preceding quadratic. Thus

$$\left(\frac{A}{H}\right)_m^2 = \frac{1}{2\sin^2 4\theta_m}\left(1 - \frac{\{\cos 8\theta_m + \frac{3}{5}\cos 4\theta_m\}}{[\sin 8\theta_m + \{\cos 8\theta_m + \frac{3}{5}\cos 4\theta_m\}^2]^{1/2}}\right)$$

where the suffix m denotes the value of the quantity when the transverse magnetization is a maximum.

Substituting this value of A/H_m in the expression for ϕ above, we have

$$\phi_m = \theta_m - \sin^{-1}\left[\frac{1}{2}\left(1 - \frac{\{\cos 8\theta_m + \frac{3}{5}\cos 4\theta_m\}}{[\sin 8\theta_m + \{\cos 8\theta_m + \frac{3}{5}\cos 4\theta_m\}^2]^{1/2}}\right)\right]^{1/2}$$

If this relation be exhibited by a curve (Fig. 25), the value of θ_m can be found from the observed value of ϕ_m. And then the

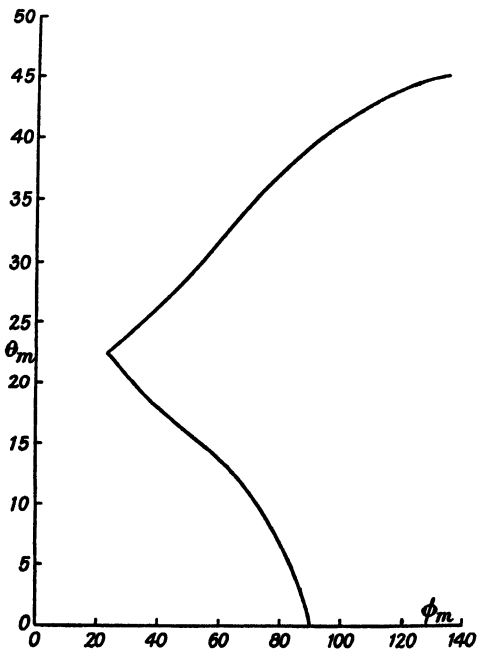

FIG. 25.

expression just given for A/H_m gives the value of A, for the crystal experimented on, when H_m has been observed as well as ϕ_m. And we can now use the curve representing

$$\phi = \theta - \sin^{-1}\left[\frac{1}{2\sin^2 4\theta_m}\left(1 - \frac{\{\ldots\}}{[\sin 8\theta_m + \{\ldots\}^2]^{1/2}}\right)\right]$$

to change curves, which represent various observed quantities in

terms of ϕ, to their equivalents in terms of θ, and so test the theoretical expressions which involved θ.

63. Test by Webster's Data for the Transverse Field in an Iron Crystal.—In his theoretical treatment of the question Webster follows Weiss's procedure regarding the molecular field. It is presumed to be a demagnetizing field proportional to I; and its components, parallel to the co-ordinate axes (cube edges) in a face plane, are expressed as proportional to a function of the inclination of the magnetization to these axes, respectively. They are thus

$$Qf(\cos\theta), \quad Qf(\sin\theta)$$

respectively, where Q is taken as a constant. Projection of these components transversely to I therefore gives

$$T = Q\,[\sin\phi f(\cos\theta) - \operatorname{ccs}\theta f(\sin\theta)]$$

Webster points out that Weiss's postulate, adopted in connection with pyrrhotite, regarding the form of f, $(f(y) = y)$, if applied to a cubic crystal, would indicate entire absence of any component of the internal field transverse to I. Therefore following Weber, Maxwell, and Weiss, in their theoretical development of the subject, he seeks to find a postulate regarding the form of f, which will fit the case of magnetite. And the form which he adopted was $f(y) = y^4$.

It is interesting to note how closely he came, by this method, to the form of the expression directly deducible from Weber's theory. He obtains

$$T = Q \sin\theta\cos\theta\,(\cos^3\theta - \sin^3\theta)$$

whereas the theory gives (§ 56)

$$T = A\sin 4\theta = 4A\sin\theta\cos\theta\,(\cos^2\theta - \sin^2\theta)$$

The theoretical requirement regarding f is $f(y) = y^3$. It is also of interest to note the correctness, from Weber's point of view, of Weiss's postulate that Q is a constant. Fig. 26 exhibits Webster's graph for his postulated law, the circles representing five observed points, which clearly show the influence of observational

uncertainties or crystalline irregularities. The full points represent the condition $T = A \sin 4\theta$, with $A = 163$. Webster took $Q = 620$.

His curve gives very close correspondence—possibly within experimental error. But although no special care was used in selecting the value of A, the theoretical curve suits all the observations on the whole better with the exception of the one evidently

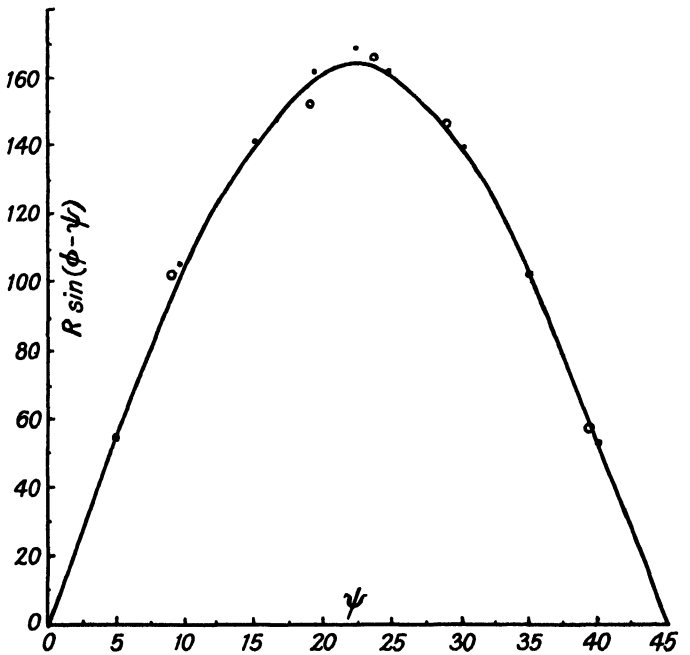

FIG. 26.

discrepant point. But the maximum of Webster's curve occurs at an angle slightly greater than 23° and the cubic symmetry necessitates its occurrence at 22° 30′. So that, even if his curve had fitted better, it could only do so on account of crystalline irregularities.

64. Test by the Variation of Magnetic Moment with Field Strength.—The closeness of correspondence between Webster's formula and the observed points show that Q may be practically regarded as a constant. In strictness, if the true

expression be $T = A \sin 4\theta$, it is not possible that Q can have the same value at all orientations of the magnetization. For the relation between Q and A is given by

$$Q = 4A \frac{\cos^2\theta - \sin^2\theta}{\cos^3\theta - \sin^3\theta}$$

Thus, with A constant, Q is variable from $4A$ to $4A$ (0·943) as θ changes from 0 to $\pi/4$. That is to say, this variation of Q does not exceed 6 per cent. Consequently, as Webster gives 10 per cent. as a possible error in Q, this source of variability will not, in practice, affect much the variation of Q with field strength.

The quantity Q, being a factor in Webster's expression for the internal field taken transverse to the magnetization, is proportional to the magnetic moment per molecule. Webster calls it the Molecular Field and measures it in terms of the gauss: and his results may be taken as giving a good experimental representation of the relation between the molecular magnetic moment M, and the field strength.

Now the intensity of magnetization is equal to nM, where n is the number of molecular magnets per unit volume. So from the postulated equation

$$I = c_0\left[H\sqrt{1 - \frac{A^2}{H^2}\sin^2 4\theta} - A\left(\frac{3}{5} + \cos 4\theta\right) - \frac{kT}{M_0}\right],$$

widened so as to take account of the molecular thermal energy in addition to the magnetic energy, as in Langevin's procedure, (§ 41), we can write tentatively

$$M = M_0 \frac{H\sqrt{1 - \frac{A^2}{H^2}\sin^2 4\theta} - A\left(\frac{3}{5} + \cos 4\theta\right) - \frac{kT}{M_0}}{H\sqrt{1 - \frac{A^2}{H^2}\sin^2 4\theta} - A\left(\frac{3}{5} + \cos 4\theta\right)} = M_0 \frac{H - B}{H - b}$$

where B and b are constants, under three assumptions. The first of these is that the approximate expression for $\coth P - 1/P$, with P large, that is, with kT/M_0H small, can be used. The second is that the term involving $\sin 4\theta$ can be neglected; but,

although Webster does not state the direction of magnetization which he worked with, he speaks of the strong opposing influence of the internal field, and this practically verifies the postulate. Finally it is assumed that, in the terms involving the longitudinal component of the internal field, M can be replaced by M_0, that is to say, A is taken as constant, although it has M as a factor. The comparatively slight variation of Q in Webster's diagram (Fig. 27) from about 480 to about 620 while H varies from 2,000 to 9,000, gives perhaps sufficient justification of this postulate (see further § 70).

The curve in Fig. 27 represents the above equation with

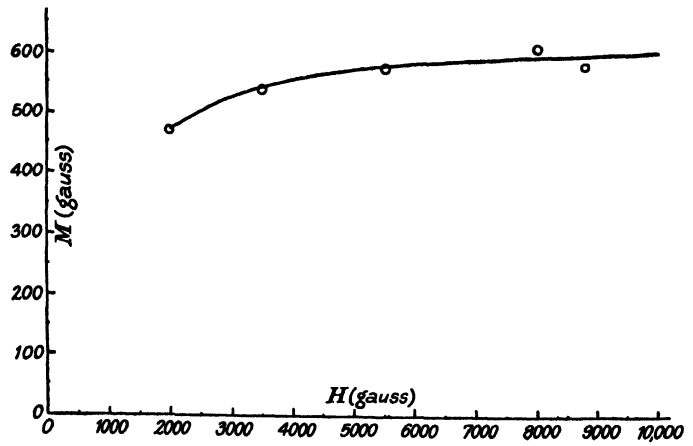

FIG. 27.

$b = 0$, $B = 554$, in Webster's units; and the circles indicate his observed values. Again the verification of the approximate theoretical expression is satisfactory.

65. Test by the Effect of Field Strength on the Maximum Transverse Component of Magnetization.—Webster also gave a curve representing the experimental connection between the maximum value of the component of magnetization, transverse to the external field, and the strength of that field. To enable us to use it as a test we have the expression

$$ {}_mI_t = c_0 A\left[\sqrt{1 - \frac{A^2}{H^2}\sin^2 4\theta} - \left(\frac{A}{H}\right)_m\left(\frac{3}{5} + \cos 4\theta\right)\right]\sin 4\theta_m $$

together with the quadratic (§ 61) for A^2/H^2 in terms of θ when I is a maximum,

$$\left(\frac{A}{H}\right)^4[4\cos 4\theta \sin 4\theta + \{(\tfrac{3}{5} + \cos 4\theta)\cos 4\theta - \sin^2 4\theta\}^2]\sin^2 4\theta + \left(\frac{A}{H}\right)^2[\text{same bracketed quantity}] + \cos 4\theta = 0$$

The small circles in Fig. 28 show Webster's results. The dotted curve is theoretical, with the same maximum, and $A = 107$.

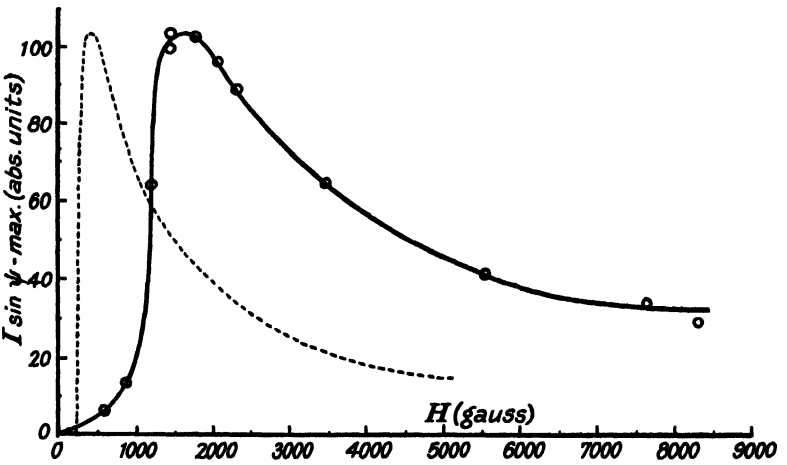

FIG. 28.

By change of the value of the A employed, which alters the scale of the abscissæ, the two maxima may be made to agree, and the general correspondence is then found to be satisfactorily close except at small fields, where the coercive force makes H finite at $I = 0$.

66. Test by Webster's Data for Transverse Magnetization.—In Fig. 29 one of Webster's sets of observations of transverse magnetization is shown. The maximum value, 96 abs., occurs when the orientation of the field, ϕ_m, has the value 18°·40′. So from Fig. 25 we find $\theta_m = 21°\cdot 50'$. And now, inserting in the expression for ${}_mI_t$ above, the value of $(A/H)_m$ in terms of

θ_m, we can calculate the value of $c_0 A$. Both factors, c_0 and A, can be found since A/H and H are known.

The values of ${}_mI_t$, at various values of θ, can now be calculated and plotted against ϕ by the help of Fig. 25. The curve obtained in this way only differs from the observed curve where the crystalline defects become evident. Crystalline asymmetry prevents

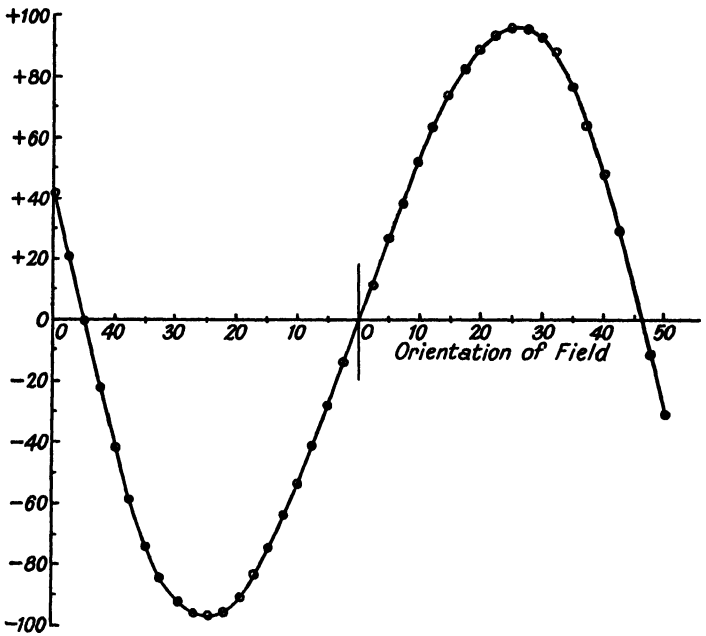

FIG. 29.

the observed effect from vanishing exactly at the ideal zero point.

67. The Magnitude of the Internal Field.—In an ideal crystal the value of A should be the same whatever may be the experimental method used in its determination. But, unfortunately, one of the most marked features of crystals is their non-homogeneity; and the effect of this is evident in the experimental diagrams. It is therefore not to be expected that quite consistent results are to be obtained from different types of observations. The fundamental question is that of the order of

magnitude of the internal field. But it is by no means impossible, *a priori*, that field magnitudes, very different from those theoretically present in a homogeneous crystal, may appear to exist in consequence of flaws.

One method available for the determination of the ratio of A to the external field, is by means of the component of magnetization parallel to the field. In the case of magnetite, the ratio of its least to its greatest value, as given by the expression for I_l (§ 58) is

$$\frac{{}_0I_l}{{}_{\pi/4}I} = \frac{1 - \frac{8}{5}\frac{A}{H}}{1 + \frac{2}{5}\frac{A}{H}}$$

From Weiss's curve (Fig. 17) this ratio has the value 808/926, whence we find

$$\frac{A}{H} = \frac{1}{15{\cdot}3}.$$

Thus the largest internal field was less than $H/6$.

Another method lies in the observation of the angle ϕ_m at which the transverse component of the magnetization is a maximum, and the value of θ_m may be found from Fig. 25. The value of A/H can then be got by evaluation of the quadratic in § 61. In this case Weiss's diagram (Fig. 17) indicates that A/H is of the order 1 to 10.

Passing now to Webster's data for an iron crystal, we find ${}_0I_l = 1{,}150$ at $H = 1{,}160$, while ${}_{\pi/4}I_l = 1{,}267$ at the same value of H. These results give $A/H = 1/20{\cdot}7$ and $A = 56$ approximately. And when $H = 1{,}450$, the observations gave ${}_0I_l = 1{,}295$, ${}_{\pi/4}I_l = 1{,}420$. So $A/H = 1/21{\cdot}7$, and $A = 67$, approximately.

The second method, used with Webster's data (Fig. 29), gives A/H between $1/11{\cdot}2$ and $1/15$, with $H = 2{,}030$. Therefore we have A between 181 and 135.

Further, the data in § 64 regarding the variation of the magnetic moment with field strength, and the condition $Q = 4A$ when $\theta = 0$, indicate, since the maximum theoretical value of the internal field is $8A/5$, values of the maximum ranging from

192 gauss at $H = 2{,}000$ gauss, through 232 gauss at $H = 5{,}000$, to 253 gauss at $H = 10{,}000$.

Thus 250 gauss is indicated as practically the saturation value of the internal field in iron.

The value of the maximum field indicated by the observations on the longitudinal component in iron at $H = 1.450$ is 98 gauss; and, at $H = 1{,}160$, the value 56 gauss is indicated.

In Weiss's observations on magnetite, the ratio of the maximum and minimum longitudinal components indicates a maximum internal field of about 100 gauss; and the angle at which the maximum value of the transverse field in magnetite occurs, indicates roughly a value of 50 gauss.

From all points of view the indications strongly point to internal fields of quite ordinary magnitudes, not exceeding, say, 300 gauss. The question of the origin of the great apparent resistance to the magnetization of pyrrhotite in the direction of its axis, and indeed also in the direction perpendicular both to the axis and to the direction of easiest magnetization, requires special consideration.

68. Forrest's Observations on Weakly Magnetic Crystals.—Dr. J. Forrest (*Trans. R.S. Edin.*, 1926) solved

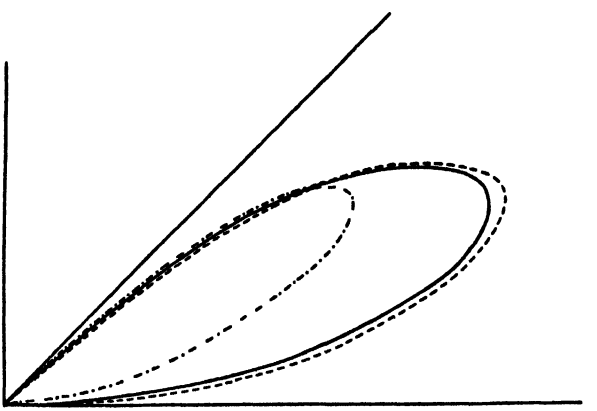

FIG. 30.

the experimental problem of applying the methods used by Weiss in connection with strongly magnetic crystals, to crystals of such weak susceptibility that they are usually regarded as

non-magnetic. Naturally in consideration of the great sensitiveness required, experimental curves cannot be expected, in this case, to exhibit so great regularity as in the other. And crystalline irregularities no greater than those observed in the former cases,

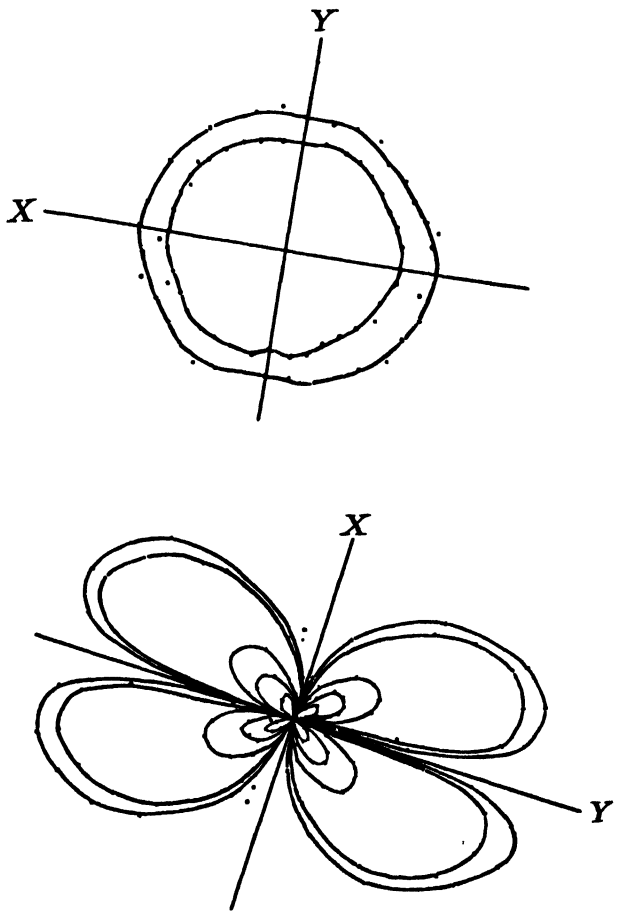

FIG. 31.

may sometimes entirely swamp the effects otherwise to be expected. The work, carried on in part on crystals other than those belonging to the cubic system, which alone we are to consider here, is shown to verify quite generally the accuracy of the approximate development of Weber's theory in its application

to crystalline structure in any system. In illustration some of his diagrams are reproduced here (Figs. 31, 32, 33).

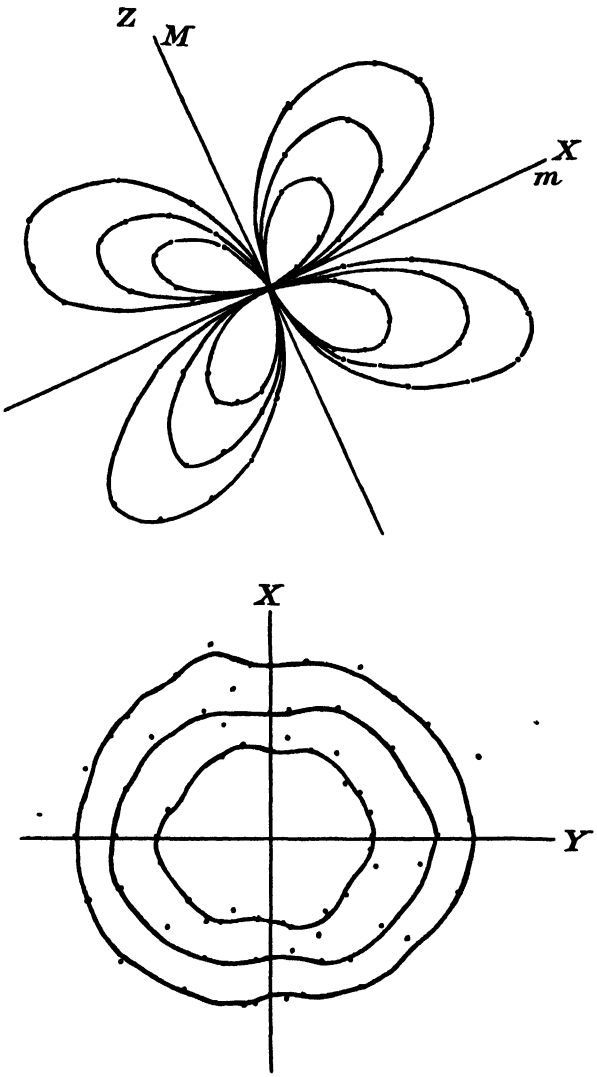

FIG. 32.

The theoretical curves shown in Fig. 30 may be compared with his observed curves of the transverse component. The constants

used in them have the values $c_0A = 100$, with $A = 156$, and $H = 313$, 2,030, 4,060 successively, in the order of increasing size of the transverse loop. Dr. Forrest concludes from his experimental results, and from his development of the theory so as to apply it in connection with the various crystalline systems, that Weber's theory "definitely predicts positions of maxima and minima therein, and in so doing enables the various possible magnetic lattices to be tested in these respects. In the majority of cases the agreement is striking, and the solution so limited that the appropriate lattice can be selected without any ambiguity." Indeed, the method was originally indicated (*Proc. R.S.E.*, 1905) as serving for the discrimination of atomic lattices before the X-ray method, really much superior from that point of view, was introduced. Similar work on weakly diamagnetic crystals of zinc and cadmium has been made by J. C. McLennan (*Nature*, 1928). It need scarcely be remarked that the magnetic lattice cannot be said, *a priori*, to be the same as the crystalline lattice. The case of iron, § 77, is outstanding in this respect.

69. Exceptionally Powerful Fields.—All the normal phenomena therefore point to the existence of internal fields, which, apart possibly from very exceptional circumstances, are of quite moderate magnitude. This, indeed, is clear merely from the consideration that it is possible to magnetize, directly and reversely at will, and in any direction, all normal magnetic crystals; and all normal bodies consisting of congeries of crystals; and this by means of comparatively moderate magnetic fields. For fields which overcome other fields must be at least as strong as they are.

Yet Weiss's observations on pyrrhotite reveal great resistance. So the question arises, Is that resistance magnetic in its origin? If so, it may be due to a change of the law of force at intermolecular distances, so that the action is much more powerful than that following from the inverse square law. This was Weiss's first idea; but, subsequently, he advanced the view that the forces were non-magnetic in their origin. If the molecular magnets also possessed coaxial electric dipolarity, the total internal field might be much greater than even a powerful external magnetic field. Surface electrification may be masked by slight conduction effects, so that the internal electric field may not be opposed by

a self dis-electrifying field. But the crystalline axis in pyrrhotit would require to be a direction in which parallelism of th

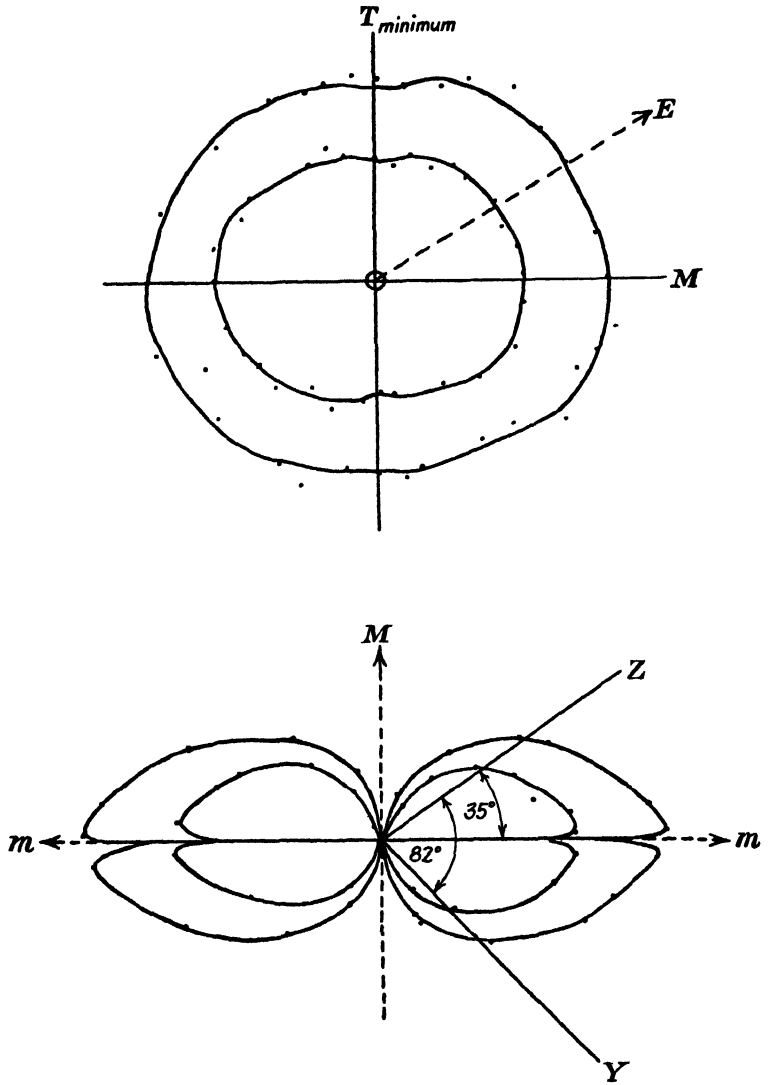

FIG. 33.

common magnetic and electric axis in each molecule would give rise to a demagnetizing and dis-electrifying field, as in the case

of a face-centred cubic crystal magnetized parallel to a cube edge. Conversely, if the electric and magnetic axes were at right angles to each other, the crystalline axis would require to be one of stable electrification. In the case of pyrrhotite, it would be more correct to speak of a plane of difficult magnetization, with a direction of easy magnetization perpendicular to it. In a rhombohedral crystal, which Weiss takes to be the elementary constituent of the quasi-hexagonal crystal, this condition can occur (*Proc. R.S. Edin.*, 1912).

In either case Weber's method still applies, though the meaning of $M52$ is altered; but, in the general application of the theory, no modification is needed.

CHAPTER VIII

TEMPERATURE AND MAGNETIZATION IN CRYSTALS

70. General Influence of Thermal Motions.—In Chapter IV we have already discussed the application, by Langevin, of thermal considerations to the case of magnetization of paramagnetic gases or solutions; and also the application by Weiss, of Langevin's procedure, to the case of paramagnetism under the influence of a random internal field. And in § 64 we have also used the method in connection with the magnetic moment as dependent on field intensity.

If we adopt no other postulate than that of the rigid ideal representative magnet, temperature can have no influence on the magnetization curves other than one of scale, for the magnetic moment of the molecules comes in only as a factor in every case. If we make assumptions which imply an influence of the crystalline structure, apart from the temperature, very different results may ensue. Avoiding these, we have to inquire into the implications of the formulæ in connection with the temperature effects.

Now Langevin's procedure involves the determination of a law of distribution of moments amongst the individual molecules; and this was found by the consideration that, at constant temperature, the kinetic energy per molecule is constant on the average. So if the incidence of the magnetic field alters the previously uniform distribution of magnetic axes throughout space, any excess of kinetic energy on the average due to greater crowding of axes within given small limits of orientation relatively to the field must be balanced by an equal withdrawal of precessional kinetic energy introduced by the action of the field. This expresses conservation of energy. But no other implication is involved. There is, for example, no condition laid down as to equipartition or otherwise of energy between molecular translational freedoms and rotational freedoms of the constituent part of a molecule wherein the magnetic property resides.

In his treatment of paramagnetism, subject to the addition, to the external field, of a randomly uniform internal field, Weiss used the same procedure. And it has also been used (§ 64) in obtaining an expression by means of which Webster's observations respecting the influence of the field strength on the intensity of magnetization of an iron crystal might be tested from the point of view of theory. We must now consider the latter treatment a little more closely, for the result throws light on some points which have arisen.

71. General Thermal Influence (continued).—In the expression for the intensity of magnetization

$$I = I_0\left(\coth P - \frac{1}{P}\right)$$
$$P = \mu H/kT$$

the extreme cases of P small and P large possess simplicity. The former corresponds to the initial, steep, and very rectilinear part of Langevin's curve (Fig. 14); the latter corresponds to the approximately rectilinear part of the curve of low slope, tending asymptotically to a maximum.

In the former case the intensity becomes

$$I = I_0 \cdot \frac{\mu H}{3kT}$$

which expresses Curie's Law, and is strongly confirmed by observation on substances in which no resultant internal field exists. And the extension of that case by Weiss, so as to include consideration of an internal field which is proportional to the intensity and acts in the same line, gives

$$I = I_0 \frac{\mu(H + NI)}{3kT}$$

and leads to

$$I = I_0 \frac{\mu H}{3k(T - \Theta)}$$

and expresses Weiss's Law. It differs from Curie's Law merely

in the respect that Θ in the latter is the absolute zero of temperature. And it, equally, has strong confirmation by observations on substances, even solids, which are paramagnetic, but do not exhibit crystalline quality in their magnetization.

Further, calculation (§ 43) with the observational data for actual substances shows that we are compelled to use a small value of P in these cases, at all ordinary temperatures, if the theoretical expressions are to be applicable.

The opposite condition held in the application of Langevin's curve (§ 64) to the explanation of Webster's data which show the dependence of the internal field in an iron crystal on the magnitude of the external field. In that case, in order to obtain correspondence with Langevin's highly verified theoretical curve we were just as strongly compelled to select the part in which P is large, and then equally good correspondence between theory and experiment followed (Fig. 27).

72. The Question of Equipartition of Energy.—We must now consider the physical implication, involved in Langevin's formulæ, regarding the proportion of the molecular kinetic energy which is converted into potential energy in the magnetic field. The point of interest in this matter is the question of the limiting value. At the one limit, when the temperature is so high, or the field is so feeble, that the ratio of M_0H to kT is small, the condition is

$$\frac{I}{I_0} = \frac{M}{M_0} = \cos\chi = \frac{M_0H}{3kT}$$

where χ is the average angular deviation of the magnetic axis from the direction of the field, and the average work done by the collisions in scattering the axes from alignment with the field is, per molecule,

$$M_0H(1 - \cos\chi) = M_0H\left(1 - \frac{M_0H}{3kT}\right)$$
$$= 3kT \cdot P(1 - P)$$

There is here no outstanding characteristic of the fraction,

$P(1-P)$, of the average kinetic energy of translational motion of the molecule. It cannot exceed one half.

At the other limit, when the temperature is so low, or the field so strong, that the ratio of kT to M_0H is small, the condition becomes

$$\frac{I}{I_0}=\frac{M}{M_0}=\cos\chi=\frac{H_1-\dfrac{kT}{M_0}}{H_1}$$

where the suffix is added to H to indicate that any internal component of the field, parallel to the direction of magnetization, which may exist, is included. And now we obtain

$$M_0H_1(1-\cos x)=kT$$

Thus the result indicated by the theory is that the average work performed, per molecule, against the field, by the molecular impacts is equal to the energy associated at the given temperature, with one degree of freedom. *This is the well-known law of equipartition of energy amongst the degrees of freedom.* It must be remembered, however, that the theory of Langevin and Weiss virtually assumes that the molecule as a whole is the seat of the magnetic quality. The conditions may be different if the magnetic quality is attached to the molecule through a constituent part of itself alone.

73. The Magnetic Constituent of the Molecule, and "Temperature."—The value of A/H in the experiments referred to in § 64 is indicated in § 67 as being of the order $1/15$. Therefore, in the tentatively adopted expression for M, we may replace the quantity under the sign of the square root by unity approximately, and so get

$$\frac{M}{M_0}=\frac{H-\left[A\left(\frac{3}{5}+\cos 4\theta\right)+\dfrac{kT}{M_0}\right]}{H-A\left(\frac{3}{5}+\cos 4\theta\right)}=\frac{H-B}{H-b}$$

in which, according to the numerical data used, $B=554$,

$b = 0$. Though these data give a larger estimate of A than any of the other methods considered in § 67, it must be remembered that they are the most uncertain of the data supplied; and the larger estimate is still of the same order of magnitude as the others. We find

$$\frac{kT}{M_0} = 293$$

Using the same numerical values of the quantities as we employed in § 43, this gives approximately

$$T = 0^\circ \cdot 045$$

on the absolute scale of temperature.

But, as already remarked, the procedure involved in the formation of Langevin's equation practically postulates that the magnetic freedom is subject to equipartition of energy with the translational freedom of the molecule. And it is well known that the magnitude of the specific heat of iron would, if there were equipartitioning amongst all effective freedoms, indicate that the molecule acted as a rigid body. The indicated "temperature" is nearly as low as one seven-thousandth part of the actual molecularly constituted temperature. If we are to retain the usual meaning of temperature, we must therefore regard k in the above formula as being numerically equal to only about one seven-thousandth part of the k of the kinetic theory.

It is of interest to note that the mass of eleven electrons, which are regarded as contributing to the magnetic quality of the atom of iron, is, even if we take their slow-speed inertia alone, about one nine-thousandth part of the mass of the atom. The electrons seem to act as if they received their proper mass-proportion of collisional molecular energy, and share it with the "magnetic" freedoms.

The complete linkage of all the phenomena of magnetism and heat which is furnished by the formula of Langevin, and its extension by Weiss, is impressive.

When the molecules of a substance are, on the average, so far apart that they exert no appreciable intermolecular magnetic action on each other, even at ordinary temperatures, M_0H/kT is

so small that the consequent approximate formula exhibits all the phenomena of paramagnetic bodies; and Curie's law follows. When they are so much closer that intermolecular magnetic forces are appreciable, but when yet the thermal motions prevent the appearance of crystalline dominance, Weiss's extension of

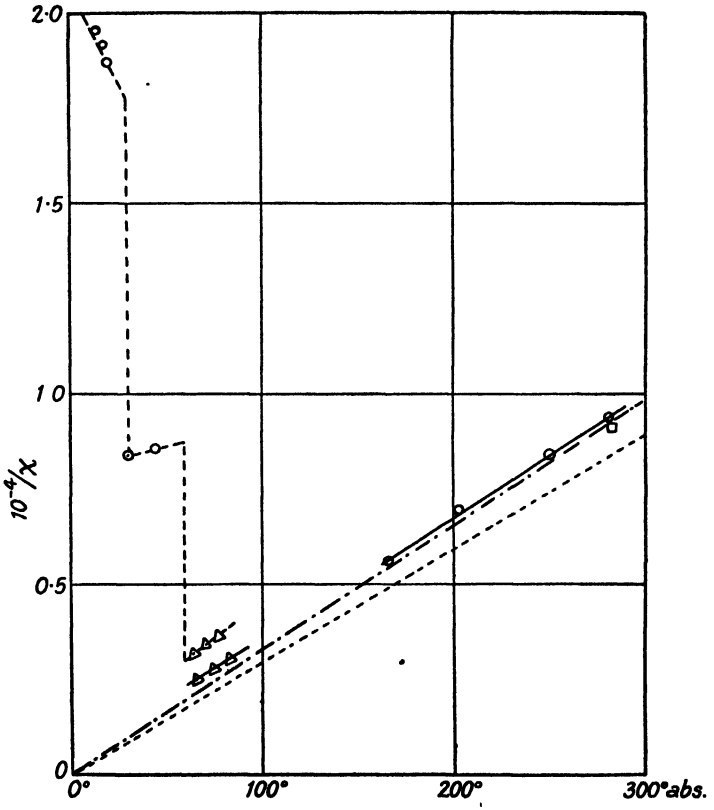

FIG. 34.

the formula becomes applicable, and Weiss's law replaces Curie's law, so long as the temperature does not fall below the Curie point.

It is quite conceivable that, in some substances, changes of molecular configuration, resulting in changes of the magnitude of the internal field, may occur at temperatures below the first Curie point without exhibition of the influence of crystalline

form. A succession of Curie points might then occur, as in oxygen in various states, see Fig. 34 (given by Stoner).

But ultimately, when the temperature becomes so low that the random thermal motions do not prevent the crystalline domination of the internal field, the normal deviation of the magnetization from coincidence with the direction of the external field comes into play, and the phenomena are predictable, in at least their general aspect, from a knowledge of the crystalline form and the application of Weber's postulates. And the experimental conditions then indicate that, before saturation is reached, a change has taken place within the molecule, or between it and its surroundings, of such a nature that the "molecular magnet" has effectively become detached from the bulk of the molecule, in so far as transformation of molecular heat energy into magnetic energy is concerned. Each magnetic element has been bound, in so far as the entire magnetic quality of the solid comes into consideration, into the single framework of the so-called "magnetic lattice," and the magnetic energy of the complete lattice has become equipartitioned with the thermal energy of the magnetic elements. When the lattice is destroyed by elevation of the temperature of the crystal, the magnetic element of the atom becomes an integral part of the atom itself, bound to it so closely that the magnetic energy stored up in it, relatively to an external field, is equipartitioned with the atomic kinetic energy itself.

If we use the formula of Langevin to define a scale of temperature by means of its expression for the fractional molecular magnetic moment in unit field, we find that the scale is the absolute scale for all temperatures when no internal field exists. When an internal field exists, but only in co-direction with the external field, the temperature is on the absolute scale reckonable from the Curie point. And, in general, the expression

$$\coth \frac{\mu H}{kT} - \frac{kT}{\mu H}$$

would furnish the scale by which the absolute temperature is given, μ, k, and H being fixed. But, when the internal component of the field is dominated by crystalline structure, the temperature indicated is excessively low in comparison with the normal temperature of the substance. The thermometer has

changed its nature and, if T is to be the true temperature of the crystal, k must be made greatly smaller than its "gas-constant" value. Forms of potential energy other than the magnetic form demand their share.

One other interpretation is possible. The observations on crystals may agree with the limiting value of the expression above, when $\mu H/kT$ is large, neither because T is very small, with k normal, nor because k is small with T normal, but because H, which includes the internal field, is large. This was Weiss's original postulate of an enormous value for the internal field, framed to explain the great insusceptibility of pyrrhotite in certain directions. But the adoption of Weber's law of interaction, and its application to crystals, leading to results in consonance with observation, shows that powerful internal fields do not exist in general. They could only be deducible from Weber's ideas under a law of force much more powerful than that of the inverse square. But the inverse square law, as used by Weber, is found to give good agreement with observation in general. Therefore any evidence of abnormal internal fields must, if it be real, be ascribed to other action, such as electrostatic action by electric dipoles. This was Weiss's final view.

74. Variation of Volume at Constant Temperature. The Coercive Force.—Further evidence on the above point can be obtained by considering the magnetic state to be varied by change of volume instead of by change of temperature.

In Langevin's curve, Fig. 14, the straight line drawn through the origin ceases to cut the curve if its slope be greater than $1/3$. If the slope has a smaller value, and if the field be entirely internal, the ordinate at the point of intersection of the straight line and the curve gives the value of the spontaneous magnetization. And if the point of intersection be near the origin, the condition is (§ 71), with $H = NI$,

$$\frac{I}{I_0} = \frac{\mu H}{3kT} = \frac{I_0 NI}{3RT} \cdot \frac{m}{\rho}$$

Therefore

$$1 = \frac{I_0^2 N}{3RT} \cdot \frac{m}{\rho}$$

is the condition that spontaneous magnetization should just

cease. If the temperature be higher than the value $I_0^2Nm/3R\rho$, the Curie point, the spontaneous condition cannot occur. In this way Weiss established his modification of Curie's law.

By using the modified law as an equation for T, taken as of normal magnitude, the internal field has the value

$$NI = \frac{3R\rho T}{mI_0^2}I$$

which is very large. We really make an assumption if we assert that, when work against the internal field is involved, T in this equation is the temperature of the whole substance. As stated above, we may test the validity of the postulate by taking the volume of unit mass as the variable on which the intensity of magnetization depends. Thus we get

$$v = \frac{1}{\rho} = \frac{3RT}{I_0^2Nm}$$

Now, T being constant, if the linear scale of the magnetic lattice be altered in the ratio $p : 1$, the density changes in the ration $1 : p^3$, and so also does the intensity, I_0, which is proportional to it. Therefore, if the formula is applicable, the maximum internal field NI_0 must be independent of the density, a very inadmissible result. In addition to all the indirect evidence of the comparative smallness of the internal field, which will be added to shortly, we have the experimental evidence of the smallness of the coercive force in iron. For the coercive force is simply the residual internal field—the force maintaining spontaneous magnetization when the external field is withdrawn.

75. The Evidence of Nickel Crystals.—The properties of single crystals of nickel are described by S. Kaya and Y. Masiyama (*Nature*, 1927), and also by Sucksmith, Potter, and Broadway; the curves which the latter workers obtained for the parallel and transverse components of magnetization, and for the variation of the latter with field strength, are of exactly the same type as those for iron, though with a fundamental difference. At a field of 150 gauss the intensities of magnetization in the

directions of the body and the face diagonals are respectively 345 and 365. Hence

$$\frac{{}_0I_l}{{}_{\pi/4}I_l} = \frac{1 - \frac{8}{5}\frac{A}{H}}{1 + \frac{2}{5}\frac{A}{H}} = \frac{345}{365}$$

and we find $A = 4 \cdot 17$.

The direction of magnetization in which the transverse com-

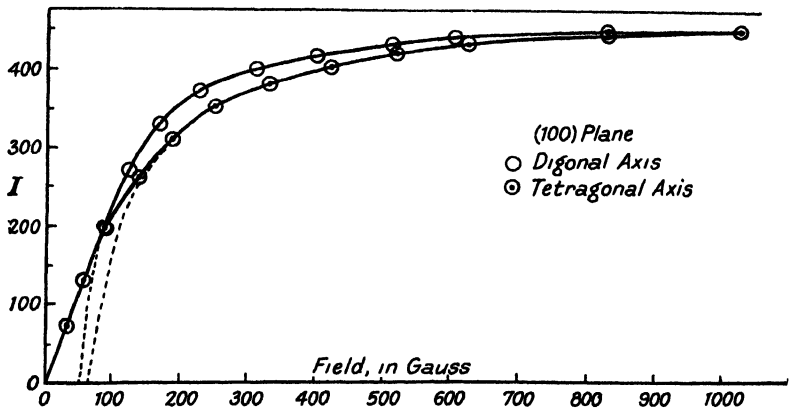

FIG. 35.

ponent is a maximum is inclined at about 22° 15′ to the unstable axis and occurs at $H = 246$ gauss. Hence the quadratic for A^2/H^2 leads to the result $A = 4 \cdot 33$ which is in full agreement with the other estimate. This indicates a maximum internal field of about 7 gauss, which is closely concordant with the observed magnitude of the coercive force in nickel.

The same observers give curves of transverse magnetization of nickel at various field strengths. These closely resemble Webster's curve for iron (Fig. 27). They give also diagrams representing the relation of total magnetization and field strength. Their results for magnetization along an edge and a diagonal in a face plane are shown by the circles and full line curves in Fig. 35, the upper set, where any difference is exhibited, refer to the face

diagonal. The dotted curves have, according to our present postulate, the equation (§ 73)

$$\frac{I}{I_0} = \frac{H\sqrt{1 - \frac{A^2}{H^2}\sin^2 4\theta} - A\left(\frac{3}{5} + \cos 4\theta\right) - \frac{kT}{M_0}}{H\sqrt{1 - \frac{A^2}{H^2}\sin^2 4\theta} - A\left(\frac{3}{5} + \cos 4\theta\right)}$$

with $\theta = 0$ for the edge, and $\theta = \pi/4$ for the diagonal. It is to be noted that the equation makes I vanish at a finite value of H. This is a necessary consequence of the demagnetizing action of the thermal motions. On the other hand, the experimental observations on single crystals of iron and nickel seem to agree in indicating a very nearly linear decay of magnetization as the field diminishes towards zero, and also a simultaneous vanishing of both. The explanation of this discrepancy will be considered later (§ 79).

Meanwhile, it is desirable to note that the equation just given does not accurately represent the consequence of our postulate regarding the partitioning of energy between the magnetic and the thermal actions. Suppose that the temperature is initially zero; spontaneous magnetization can occur in any one of the stable directions of magnetization. And if an external field were directed along one of these, and were sufficiently strong, the residual magnetization when the field was withdrawn would be in that one direction alone, and would have its saturation value at zero temperature. If the internal field is directed oppositely to the external field and to the direction of magnetization, no residual magnetization can exist in that direction even at zero temperature. In the intermediate stage work has been done against the external field, by the internal field, or by the molecular impacts when the temperature differs from zero, to the amount

$$HM_0(1 - \cos\phi)\cos\theta$$

per molecule on the average, where ϕ is the inclination of the axes to the direction of magnetization, while θ is the inclination of H to I. Similarly the internal field having the component

$A\,(\frac{3}{5} + \cos 4\theta)\cos\phi$ in the direction of magnetization, the work done against it as the temperature rises from zero to T is

$$-A\,(\tfrac{3}{3} + \cos 4\theta)\int_0^{\phi} \cos\phi\, d(M_0 \cos\phi)$$

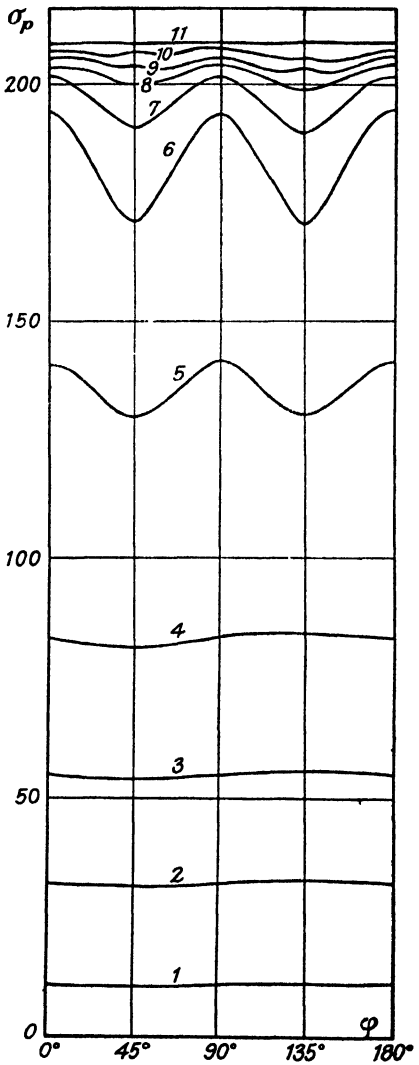

FIG. 36.

Therefore its value is

$$M_0 \frac{A}{2}(\tfrac{3}{5} + \cos 4\theta)(1 - \cos^2\phi)$$

So the whole work done by the thermal impacts is the sum of

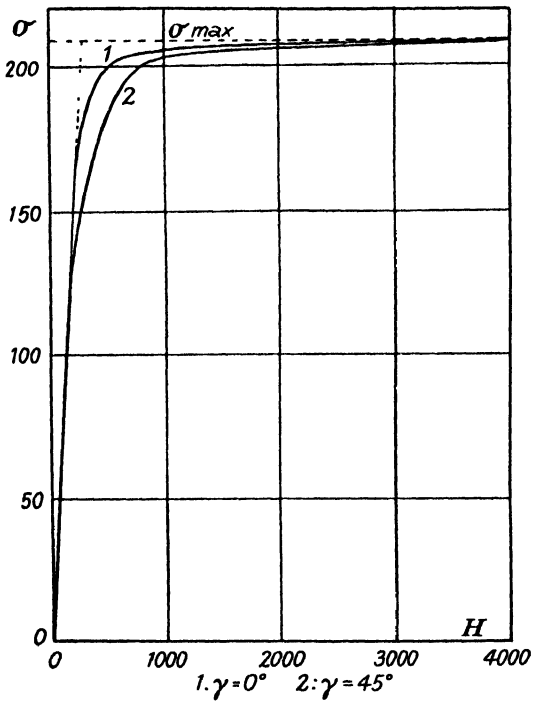

FIG. 37.

these quantities; and, if it is equal to the thermal energy per degree of freedom, we get

$$H\cos\theta(1 - \cos\phi) + \frac{A}{2}(\tfrac{3}{5} + \cos 4\theta)(1 - \cos^2\phi) = \frac{kT}{M_0}$$

This is the equation giving the relation between H and I at any given temperature, in any given direction of magnetization, for the magnets having the inclination ϕ. It still indicates residual

magnetization; and its further discussion will be taken up when we deal with the phenomena of hysteresis (§ 78).

76. Beck's Work on Iron Crystals.—The work of K. Beck on crystals of iron preceded that of Webster, and was very elaborate. Webster's results agreed with his in all essential features.

Fig. 36 represents his observations on one plate cut parallel to a face plane of the cubic crystal, with reference to the parallel component of magnetization in various directions in the plane of the plate.

Fig. 37 is his diagram, for the same plate, showing the dependence, upon the field strength, of the magnetization, I_b and I_q, in the directions of the binary and the quaternary axes, respectively.

The reference for Beck's work is: *Viert. Natur. Gesellsch., Zurich*, t. 63, 1918, p. 116. The work of W. L. Webster on the subject is published in the *Proceedings of the Royal Society*, Vol. 107, 1925.

77. The Magnetic Lattices in Nickel, Magnetite, Iron, and Heussler Alloy.—A comparison of Beck's and Webster's diagrams for iron with those of Sucksmith, Potter, and Broadway for nickel, and Weiss's diagrams for magnetite, make clear two fundamentally different characteristics of iron relatively to the others. The maximum susceptibility for iron is in the directions of the quaternary axes, and the minimum is in the directions of the binary axes. The opposite holds true with regard to nickel and magnetite. Again, the maximum value of the transverse magnetization in iron is in a direction more nearly coincident with the binary than with the quaternary axes, and the reverse is true of the other two substances. This indicates a difference of crystalline structure. In the former, A is positive, for the directions of stable magnetization are those of the quaternary axes; in the latter cases A is negative, the directions of stability in the face plane being the binary directions.

Now the summations for the evaluation of the internal field show that A is positive when the magnetic lattice is simple cubic, while it is negative when the magnetic lattice is either the face-centred or the body-centred lattice. But X-ray analysis indicates

that all three substances have either a face-centred or a body-centred lattice.

Therefore the Weberian analysis gives us the clear indication, as a result of direct observations of magnetic quality, that the magnetic and the molecular lattice are of identical type in magnetite and nickel; and it as distinctly shows us that, in iron, the magnetic lattice differs in type from the molecular lattice. In this case the magnetic analysis goes deeper than the X-ray analysis, and gives us information regarding the atomic structure, which is not indicated by that method.

Potter has recently investigated crystals of Heussler alloy, and has found that the magnetic properties are almost identical with those of nickel. Now X-ray analysis shows that the atomic lattice of the alloy is that of a body-centred cube, while that of nickel is face centred. This is in entire agreement with theory if the magnetic lattices correspond with the atomic lattices. A careful quantitative comparison of the magnetic curves relating to the two substances is necessary for the settlement of the question. Great difficulty may possibly be found if crystalline irregularities occur, and also because of the demagnetizing effect of the boundaries of the specimens.

The work of Sucksmith, Potter, and Broadway, on crystals of nickel, is detailed in the *Proc. Roy. Soc.*, A, 1928. That of Potter, on Heussler alloy, appears in the *Proceedings of the Physical Society*, Feb. 1929.

78. Cylical Phenomena. Hysteresis.—The equation given at the end of § 75 is

$$H\cos\theta\left(1-\frac{I}{I_0}\right)+\frac{A}{2}\left(\frac{3}{5}+\cos 4\theta\right)\left(1-\frac{I^2}{I_0^2}\right)=\frac{kT}{M_0}$$

if we replace $\cos\phi$ by the ratio of I to I_0; and it gives

$$\frac{I}{I_0}=-\frac{H\cos\theta}{H_1}+\sqrt{\frac{(H\cos\theta+H_1)^2-2H_1\frac{kT}{M_0}}{H_1^2}}$$

where $H_i = A(\frac{3}{5} + \cos 4\theta)$ is the saturation component of the internal field in the line of magnetization, H_i being taken as positive in the direction of magnetization. The meaning of the equation can be readily seen by putting $H = 0$, $T = 0$, in which case

$$\frac{I}{I_0} = \pm 1$$

This signifies saturation under the internal field alone, in either the positive or the negative stable direction.

Suppose next that, still with $T = 0$, the external field is applied with I negative. No change takes place in I until $H \cos \theta = H_i$, when the magnetization changes from $-I_0$ to $+I_0$. If the field be reversed gradually, there is again a sudden reversal of magnetization when $H \cos \theta = -H_i$, after which the field may be reduced to zero with no change in the magnetization. A complete rectangular loop of magnetization—a hysteresis loop—has been described.

Now let T have a finite value. Two corners of the rectangle become rounded, the other pair are displaced outwards in the directions of increasing field strength, the rectilinear sides become curved, and we obtain a normally shaped hysteresis loop. If we postulate an initial condition of zero resultant magnetization, which can readily occur because of crystalline irregularities, but may also take place in a homogeneous crystal, the effect which appears as H is increased from zero will depend on the values of H_i and T. If T be relatively small enough, the magnetization will rise from zero, in the well-known manner, exhibiting an initial slow rise, and an intermediate rapid rise, with a final slow asymptotic approach to saturation.

If the line of attempted magnetization be one of instability, the sign of H_i is opposite to that of I. Saturation is only approached when the external field is very large, and, as the field is lessened, the curve of magnetization is retraced throughout its whole course. Or, rather, this is what would occur if magnetization could be prevented from taking place along one of the lines of stable magnetization parallel to which there is a component of the externally acting field.

79. Retentivity. The Non-Retentivity of Iron and Nickel Crystals.—If the law postulated at the beginning of last section be valid, magnetization can only take place if

$$(H\cos\theta + H_i)^2 \lessdot 2H_i\frac{kT}{M_0}$$

And, if there be no external field, there can be no spontaneous magnetization unless

$$H_i \lessdot 2\frac{kT}{M_0}.$$

Thus if, at ordinary temperatures, iron and nickel crystals really do not exhibit retentivity, a condition which the experimental results of various observers seem to indicate (see Figs. 25, 37), an explanation might be based on the smallness of k at these temperatures.

Since the magnetic lattice of iron is simple cubic, the cube edges are directions of stable magnetization, and the body diagonals are directions of magnetization which give the strongest reverse field internally. Thus with $\cos 4\theta = 1$, we have

$$H_i = A(\tfrac{3}{5} + \cos 4\theta) = \tfrac{8}{5}A$$

$$A = \frac{2M_0a^2}{\rho^5}(D - G)$$

the quantities being identical with those used in Chapter VI. The value of $D - G$, reckoned over ten molecular distances, is $67\cdot3$; and the quantity to be added to this on account of the remainder of the molecular magnets, $4\pi I/3$, with $I = 1{,}700$, is 1,134. This gives 1,200 as the total value of $D - G$.

Hence, using $M_0 = 1\cdot8(10)^{-20}$, $k = 1\cdot4(10)^{-16}$, $\rho = 2(10)^{-8}$, and taking $a = 5(10)^{-9}$, we find

$$T \gtrdot 15^\circ \text{ abs.}$$

if residual magnetization is to appear. In the calculation the full kinetic theory value of k is used. If the actual value of k in the

magnetic lattice were one-twentieth of the full value, retentivity should just be exhibited at ordinary temperatures; and, if it were so small as one-fiftieth, a single magnetized crystal of iron should always show retentivity.

In support of the view that k may be small at ordinary temperatures we have, in addition to the evidence already referred to, the proof by Weiss and Onnes that the spontaneous magnetization of iron at 20·3 abs. is only 1·021 times greater than its value at 20° C. Other indications will be given in § 80 and § 81.

80.—Temperature Variation of Spontaneous Magnetization.—The equation at the commencement of § 78 gives, as the law of spontaneous magnetization, the expression

$$\left(1 - \frac{I}{I_0}\right)^2 = \frac{T}{\frac{A(\frac{3}{5} + \cos 4\theta)}{2k}} = \frac{T}{\Theta}$$

where Θ, the temperature at which the spontaneous magnetization vanishes, is dependent on θ. In iron, as we have repeatedly seen, its maximum is in the direction of a cube edge.

The other tentative formula used in § 64, if applied to this problem, leads to

$$1 - \frac{I}{I_0} = \frac{T}{-\frac{A(\frac{3}{5} + \cos 4\theta)}{k}} = \frac{T}{\Theta}$$

The difference in sign is merely due to the selection of the opposite sign for A with $\theta = 0$. The temperature Θ is necessarily positive, so that the sign of A and the value of θ for the direction of spontaneous magnetization must be appropriately chosen. But the value of A/k given by the experimental determination of Θ would, in the latter case, be half of its value in the former case. Experiment decides at once in favour of the former, as a glance at Fig. 38 will show. For the latter gives, as the curve of spontaneous magnetization, the straight line joining the points (1, 0) and (0, 1), which is much less nearly in accordance with observation than the curve indicated by the other formula.

The full line in the figure gives the result corresponding to the assumption that, as in Langevin's treatment of paramagnetism,

Boltzmann's law of the distribution of the molecular magnetic axes holds. The lower dotted curve expresses the first formula above, and shows that the postulate regarding axial distribution which is involved in it is not so good as that involved in Boltzmann's treatment. The upper dotted curve is in much better agreement with observation than the full line curve, and expresses the empirical condition

$$1 - \left(\frac{I}{I_0}\right)^4 = \frac{T}{\Theta}$$

But even it is defective in the region of approximate saturation;

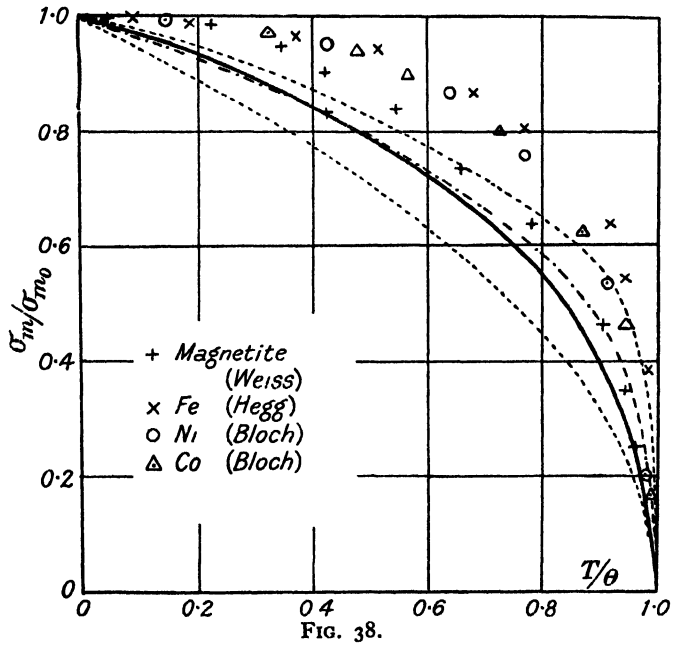

FIG. 38.

in fact no power law will fit the experimental results well throughout the whole range.

The significance of the data is great. *In the region of very low temperature, saturation is almost complete over a considerable range.* This means that the k in the formula given at the commencement of § 78 is much smaller than the k of the kinetic theory which refers to free molecular impacts. *In the mechanism of the magnetic lattice there is screening from the full effect of the*

thermal molecular motions. If there were complete screening, k would be zero, and the curve of magnetization in Fig. 38 would be the line of saturation till the temperature, Θ, of dissolution of the screening influence was reached, when the thermal motions would suddenly attain full efficiency, and spontaneous magnetization would be impossible.

An outstanding problem for settlement is the determination of k as a function of T/Θ, or I/I_0.

81. Other Theoretical Work.—An interesting case occurs if we presume that the trend of k is such that we may replace M_0 in the formula by $M_0 \cos^2 \phi$. This means that the hitherto postulated "molecular magnet" is not an ideal rigid magnet, but is composite. The dot-dash line in Fig. 38 shows that the results regarding spontaneous magnetization which follow from its use agree better with observation than those given by the Langevin-Weiss curve (full line). The outstanding feature is the establishment of a thermomagnetic equation of state having close analogies with the well-known thermomechanical equation of state (*Proc. R.S.E.*, 1927).

The pioneer in this direction was Hitchcock, whose papers should be consulted (*Phil. Mag.*, 1912, *et seq.*).

Other theories, like the modification just alluded to, postulate special structures for the molecular magnets. The work of Honda and Okubo (*Sc. Rep. Tokyo Univ.*, 1916) is of this type. So also is that of Gauss (*Ann. der Physik*, 1916), which is very instructive.

Frivold (*Ann. der Phys.*, 65, 1921) has investigated the case of a random crystalline aggregate, constituted of cubic crystals, with special regard to the rapid rise of magnetization at low fields which is characteristic of ferromagnetism. He points out that the proof, because of the impossibility of effecting full integration, is not quite complete. But, in the cases in which evaluation was carried out, the indication is that the characteristic of ferromagnetism is not accounted for by the action of the internal field, not only if it be regarded as due to Weber's mutual actions of the magnets, but even if electrostatic aid, originating in co-existing electric dipoles, be present (§ 69).

His procedure follows the Langevin-Weiss lines, and adopts the kinetic theory value of k. It is easy to show that, if k has only 1/1,000th part of the kinetic theory value, the characteristic

rapid rise of the magnetization of iron in low fields would be evident.

This condition by itself, without aid from the other arguments already specified, points very powerfully to the smallness of the share of molecular thermal energy which is associated with the molecular magnets.

CHAPTER IX

THE QUANTUM AND MAGNETON THEORIES

82. Origin of the Quantum Theory.—The theory of equipartition of energy asserts that, in a dynamical system obeying Newton's laws of motion, and possessing freedom for intercommunication of energy amongst its several parts, the final average condition of equilibrium is one in which the total energy is partitioned equally amongst the freedoms. In order that this condition may hold at practically any instant, the system considered must possess a very large number of freedoms of each type which comes into play.

A mass of glowing gas is such a system; but, since it is radiating, the gas is in energetic communication with the ether. And so we are forced to consider, say, the gas and ether within a finite, closed, perfectly reflecting boundary. But, even although it be finite, the volume of ether concerned has a practically infinite number of freedoms corresponding to the different modes of vibration which can appear—just as an ideal string supposed to be continuous in its structure and perfectly flexible, has an infinite number of freedoms corresponding to the various harmonic modes of vibration of which it is capable. Therefore the doctrine of equipartition asserts that the gas and the finite volume of ether in radiative communication with it, must constantly tend towards a final state in which most of the energy appears in connection with the high-frequency vibrations of the ether.

It is possible to calculate the equipartitional distribution of energy in the spectrum of a perfectly radiating body; and it is found to be greatly different from the observed distribution, especially at short wave-lengths. Concordance with observation is found by denying the postulate of continuous emission and absorption of radiation by matter. Emission is not regarded as possible until the conditions are such that definite amounts of

energy are emitted. These amounts, in the case of radiation of frequency ν, are typified by

$$E = nh\nu$$

where n is an integer and h is an absolute constant—one of the so-called "constants of nature."

Since ν is a frequency, that is, the reciprocal of a time interval, the dimensions of h are those of energy and time conjointly. In other words, its dimensions are those of a time-integral of energy, or action, as it was called in classical dynamics. Processes in nature which were carried out in accordance with the laws of classical dynamics were such that the action in proceeding from a given initial state to a given final state, was a minimum, or at any rate stationary. In quantum dynamics h is found to be an absolute unit or quantum of action, and its physical reality is made evident by its constancy as determined by observations and measurements made in many diverse fields of physical investigation—including magnetic.

83. Quantum Electronic Orbits.—The modern view of the constitution of the atom is that it consists of a central nucleus made up of positive and negative charges of electricity, all of equal amount e, the magnitude of the electronic charge—another absolute "constant of nature." In the nucleus there is an excess of positive charges (protons) over the negative charges (electrons); and, in a neutral atom, a number of electrons equal to that excess circulate in orbits round the nucleus, as planets circulate round the sun. The central force being attractive, and depending in the same manner on the distance as in the gravitational case, the path of an orbital electron is elliptic to a similar degree of approximation as in the planetary system.

If classical dynamics were applicable within the atom, any orbit obeying the approximate Keplerian laws might exist. In Bohr's application of quantum ideas to the atomic system, certain orbits alone, satisfying certain quantum conditions, are possible. Unlike the classical conditions, which require emission in consequence of the revolution of the electrons, the quantum conditions assert that no radiation occurs while the stable orbit is being described. This at once gets rid of the difficulty of explaining

the constant maintenance of orbital frequency, as made evident by the atomic spectra, while radiation of energy proceeds. Emission is supposed to occur only when an electron leaves or enters a stable orbit. If it leaves one and enters another, a quantum of radiation of frequency ν given by

$$E_1 - E_2 = h\nu$$

is emitted or absorbed according as the orbital potential energy E_1 exceeds or falls short of the potential energy E_2.

84. The Bohr Magneton.—If we consider for simplicity a circular orbit of radius r, the kinetic energy of the electron is $m\omega^2r^2/2$, and the total energy, half kinetic and half potential, is

$$m\omega^2r^2 = nh\nu$$

by the quantizing condition. Hence the moment of momentum is

$$\mathcal{J} = mr^2\omega = nh\nu\frac{1}{\omega} = \frac{nh}{2\pi}$$

Thus *the angular momentum is also quantized.* Indeed, we may look upon this, if we desire, as the more fundamental quantization, from which the energy quantization follows.

Now, in § 18 we found that the magnetic moment of the electronic orbit is

$$M = \frac{evr}{2} = \frac{er^2\omega}{2} = \frac{e}{m}\cdot\frac{nh}{4\pi}$$

which expresses *quantization of the magnetic moment.* Therefore, if we put $n = 1$ in this expression, we obtain an absolute unit of magnetic moment

$$U_m = \frac{e}{m}\cdot\frac{h}{4\pi}$$

This is known as the *Bohr Magneton.*

In electromagnetic units its value is $9{\cdot}23(10)^{-21}$. If this be multiplied by Avogadro's number, it gives, as the magnetic

moment, per gramme-atom, for a substance composed of single quantum atoms, the value 5,593 e.m.u.

85. The Weiss Magneton.—The idea of the physical existence of a magneton, that is, a fixed minimal atomic magnetic moment, similar to the electron as a fixed minimal electric charge, was first put forward by Weiss with supporting experimental evidence. Its establishment, in so far as it may be an expression of fact, rested on an empirical basis, whereas the possible existence of Bohr's magneton was indicated by theory. Weiss found that the magnetic susceptibility of magnetite, between each of a series of ranges of temperature, was a linear function of the reciprocal of the absolute temperature, and the slopes of the representative lines were proportional to a series of small integers. He regarded this as evidence of a set of different magnetic moments, each of which was assumed by the molecule within a corresponding range of temperature. Confirmation was obtained from observations on other substances to an extent which gave considerable support to the view. Yet it could not be regarded as quite complete, for the ratios of the observed moments in some cases differed from ratios of small integers by more than the actually possible errors of observation.

The value of this magneton as finally assumed was $1 \cdot 854(10)^{-21}$ e.m.u.; and this is almost exactly one-fifth of Bohr's magneton. There is no *a priori* impossibility that direct magnetic measurements might give a different value for the magneton from that afforded by experimental tests, in very large part of optical type, which strongly support Bohr's value. The difficulty lies in the supposition that a smaller unit than Bohr's, which must therefore be the true minimal unit if it exists, can really have a physical existence. For Bohr's unit is the necessary minimal unit of the quantized electron theory, which has wide confirmation in many branches of physics, and which co-relates it with the other "natural" constants.

86. Orientational Quantization.—We have already seen how quantization of electronic orbits can be effected with regard to energy or momentum. Now the normal to an electronic orbit is co-directed with the axis of the equivalent magnet. The full magnetic moment, M_0, is evidenced when that axis is co-directed

with the external field. If these two directions are inclined at an angle ϕ, the effective moment in the direction of the field is $M_0 \cos \phi$.

This makes it clear at once that, in all the problems connected with the exhibition of magnetization by molecular aggregates, crystalline or otherwise, which we have been discussing, the question of the distribution of orientations is fundamental. For the law of magnetization is dependent thereon.

Two methods of procedure are possible, both of which are exemplified in the preceding discussions. We may postulate the law of distribution, and deduce the law of magnetization. This is the procedure in Langevin's treatment of paramagnetism. Or we may postulate the law of magnetization, which settles the law of orientation. In Weiss's treatment of the effect of the internal field, indeed, both processes are used; the distribution law is assumed, exactly as by Langevin, but its incidence is altered by an addition to the expression for the potential energy. In all the cases dealt with in the preceding chapters, orientation of the axes has taken place under the action of the thermal motions. The resulting law of distribution has been a continuous one.

In Fig. 38 the failure of the Langevin-Weiss law to do more than give a general description of the trend of magnetization as the temperature varies is evident. The question arises whether or not quantization can give aid.

The orientation law of quantization is that certain orientations are possible, and are equally probable, the possible cases being fixed by the values

$$\cos \phi = \frac{1}{n}, \quad \frac{2}{n}, \quad \frac{3}{n}, \ldots, \frac{n}{n}$$

where n is an integer, which may be different in different systems. The mean value of $\cos^2 \phi$ is therefore

$$\cos^2 \phi_m = \sum_1^n \left(\frac{p}{n}\right)^2 = \frac{(n+1)(2n+1)}{6n^2}$$

The case of $n = \infty$ corresponds to the case of no quantization, and reproduces Langevin's value $1/3$ in the expression

$$\chi = \frac{M_0^2 \cos^2 \phi_m}{kT} = \frac{M_0^2}{3kT}$$

If we put $n = 3, 2, 1$, respectively, the numerical multiplier of k here becomes $1{\cdot}93$, $1{\cdot}6$, 1, respectively. And when $n = 1$, the

axes are all orientated parallel to the field, so that $I = I_0$. But Langevin's expression for I/I_0 at very low temperatures becomes (§ 41), $1 - kT/M_0H$. Therefore the case $n = 1$ means complete saturation at very low temperatures, and so accords well with the initial trend of the curve of Fig. 38. Its subsequent course indicates a gradual inclusion of higher values of n as the temperature rises. The variation may be just as well expressed by a rise of k from zero.

87. The Experiment of Gerlach and Stern.—Gerlach and Stern carried out experiments in which a narrow stream of atoms, under conditions giving great freedom from collisions, was made to pass across an intensely non-uniform magnetic field. The stream was found to split into two portions in such a way as to indicate the existence of two discrete orientations of their magnetic axes. In this way the first direct evidence, from experiment, of orientational quantization was obtained. Further, an experimental measurement of the magnitude of the atomic magnetic moment can be found. In the case of silver, the theoretical value of one Bohr magneton was verified. In the case of nickel two Bohr magnetons were evidenced.

And it is of interest to note, in view of the possible subdivision of the Bohr magneton, that in one case (thallium) the value of one-third of a Bohr magneton was indicated.

88. The Spinning Electron.—In the account given in the earlier chapters of the origin of magnetic quality, the presumption dealt with was that of orbital revolution of the electron. Incidentally it was found that the ratio of the magnetic and mechanical momenta, deduced on this basis, was at variance with observed results. This difficulty can be made to disappear through the introduction of an independent spin of the electronic charge itself. And this idea gives aid also in related problems of other branches of physics, notably that of the series exhibited in optical spectra. But those are questions which lie outside the scope of this book: a good synopsis is given in Allen's *Quantum Theory*.

The student who desires to follow the development of Magnetic Theory more fully will derive benefit from a study of the Bulletin of the National Research Council, Vol. 3, Part 3, No. 18, 1922 (*Report on Theories of Magnetism*).

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